Describing the Free Orthomodular Lattice with Two Generators

Lucilla Blessing

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Abstract

In this thesis we outline the fundamentals of the theory of orthomodular lattices before deriving the structure and basic properties of the free orthomodular lattice with two generators. We then introduce a notation proposed by Mirko Navara for the elements of this lattice, and extend its original scope to a novel form of arithmetic in orthomodular lattices.

Contents

Chapter 1

Basic Concepts

1.1 Lattices

A *lattice* is a set L equipped with two binary operations $\wedge$ and $\vee$ (read "meet" and "join") such that both operations are commutative and associative and they satisfy the *absorption laws*

$$\forall\, a, b \in L\colon\ (b \vee a) \wedge a = a \quad \text{and} \quad (b \wedge a) \vee a = a.$$

All lattices also satisfy the *idempotent laws*: $a \wedge a = a = a \vee a$ for all $a \in L$. This is a direct consequence of the absorption laws:

$$a \wedge a = ((a \wedge a) \vee a) \wedge a = a; \qquad a \vee a = ((a \vee a) \wedge a) \vee a = a.$$

Observe that the six lattice axioms (two associative laws, two commutative laws, two absorption laws) can be grouped into pairs in which $\wedge$ is swapped with $\vee$ and vice versa. This implies that if $(L, \wedge, \vee)$ is a lattice, then so is its *dual lattice* $(L, \vee, \wedge)$. Thus we obtain ([4], p. 5):

Duality Principle for Lattices. *If a statement is true of all lattices, then its dual (obtained by exchanging $\wedge$ with $\vee$) is also true of all lattices.*

In any lattice, for any two elements $a, b \in L$, the conditions $a = a \wedge b$ and $b = a \vee b$ are equivalent: suppose $a = a \wedge b$ holds, then by the absorption laws $a \vee b = (a \wedge b) \vee b = b$; the reverse implication is analogous. We denote both of these equivalent properties as $a \leq b$. The dual notion is $\geq$, which is also the converse relation.

The relation $\leq$ thus defined is reflexive, antisymmetric, and transitive, making L into a poset: all $a, b, c \in L$ satisfy

$$\begin{gathered} a = a \wedge a; \\ (a = a \wedge b \text{ and } b = b \wedge a) \text{ implies } a = b; \\ (a = a \wedge b \text{ and } b = b \wedge c) \text{ implies } a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c. \end{gathered}$$

If the lattice $(L, \wedge, \vee)$ induces the poset $(P, \leq)$, then its dual lattice $(L, \vee, \wedge)$ induces $(P, \geq)$, which is also a poset.

Some elementary properties through which this partial order interacts with the lattice operations are as follows: for all $a, b, c \in L$,

- $a \wedge b \leq a$ and $a \wedge b \leq b$; dually $a \vee b \geq a$ and $a \vee b \geq b$; (P1)
- $(c \leq a$ and $c \leq b) \Rightarrow c \leq a \wedge b$; dually, $(c \geq a$ and $c \geq b) \Rightarrow c \geq a \vee b$; (P2)
- Isotone property: $a \leq b$ implies $c \wedge a \leq c \wedge b$ and $c \vee a \leq c \vee b$. (P3)

The proofs (shown here only for the versions with $\wedge$; the others are dual) are straightforward:

$$\begin{gathered} a \wedge b = (a \wedge a) \wedge b = a \wedge (a \wedge b) \quad \text{and} \quad a \wedge b = a \wedge (b \wedge b) = (a \wedge b) \wedge b; \\ (c = c \wedge a \quad \text{and} \quad c = c \wedge b) \quad \text{implies} \quad c = c \wedge b = (c \wedge a) \wedge b = c \wedge (a \wedge b); \\ a = a \wedge b \quad \text{implies} \quad c \wedge a = c \wedge (a \wedge b) = (c \wedge c) \wedge (a \wedge b) = (c \wedge a) \wedge (c \wedge b). \end{gathered}$$

Our thus constructed poset is even a *lattice poset*: it has an additional property, which we will now exhibit. Let P be any poset and define the *greatest lower bound* or *infimum* of a subset $K \subseteq P$, denoted $\inf K$, as an element $x \in P$ such that $x \leq a$ for all $a \in K$ and, for any $t \in P$, the condition ($t \leq a$ for all $a \in K$) implies $t \leq x$. The *least upper bound* or *supremum*, denoted $\sup K$, is defined analogously by substituting $\geq$ for $\leq$. By antisymmetry, greatest lower bounds and least upper bounds are unique if they exist. We say that a poset P is a *lattice poset* if for any subset $\{a, b\} \in P$, the infimum and supremum exists. Our poset constructed from the lattice $(L, \wedge, \vee)$ is a lattice poset, in which additionally the infima and suprema are equal to the meets and joins: $\inf\{a, b\} = a \wedge b$ and $\sup\{a, b\} = a \vee b$. This is a direct consequence of the properties (P1) and (P2).

Conversely, let us start with a lattice poset P and define two binary operations $\wedge$ and $\vee$ on P by

$$a \wedge b := \inf\{a, b\}; \qquad a \vee b := \sup\{a, b\}.$$

Then the resulting structure on P is a lattice. The following result holds ([4], pp. 12–14):

Two Definitions of Lattices. *For a lattice $(L, \wedge, \vee)$, denote by $\leq_\wedge$ the relation defined by $a \leq_\wedge b$ iff $a = a \wedge b$, and for a lattice poset $(P, \leq)$, denote by $\wedge_\leq$ the operation $a \wedge_\leq b = \inf\{a, b\}$. Define $\leq_\vee$ and $\vee_\leq$ analogously.*

Then:

- *For any lattice poset $(P, \leq)$, $\leq_{\wedge_\leq} = \leq$ and $\leq_{\vee_\leq} = \leq$.*
- *For any lattice $(L, \wedge, \vee)$, $\wedge_{\leq_\wedge} = \wedge$ and $\vee_{\leq_\vee} = \vee$.*

Thus the two definitions of lattices, either as a poset with additional properties or as a universal algebra, are equivalent.

A *lattice homomorphism* $\phi : L_0 \to L_1$ between two lattices L_0, L_1 is a map that preserves the meet and join operations, i.e. $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$ and $\phi(a \vee b) = \phi(a) \vee \phi(b)$ for all $a, b \in L_0$. An *order homomorphism* ϕ is a map that preserves the order relation, i.e. $a \leq b \Leftrightarrow \phi(a) \leq \phi(b)$ for all $a, b \in L_0$. Because the definitions of lattices as algebras and as orders are equivalent, a map between two lattices is a lattice isomorphism iff it is an order isomorphism.

For elements a, b of a lattice L, we write $a \prec b$ and say that b *covers* a (a is *covered* by b) if $a < b$ and there does not exist any $x \in L$ such that $a < x < b$. The order relation $\leq$ in a finite lattice is fully characterized by its covering relation ([4], p. 6). Accordingly, finite lattices are often represented by their *Hasse diagram*: a graph in which elements of the lattice correspond to vertices, and for any a, b there is a directed edge from a to b iff $a \prec b$ holds, drawn so that b is vertically above a. Since $<$ is the transitive closure of $\prec$ in a finite lattice, this completely characterizes the lattice. Conversely, in an infinite lattice $<$ may even be a strict total order while $\prec$ is empty, as in $(\mathbb{Q}, \min, \max)$. Figure 1.1 depicts the five non-isomorphic nonempty lattices with up to four elements.

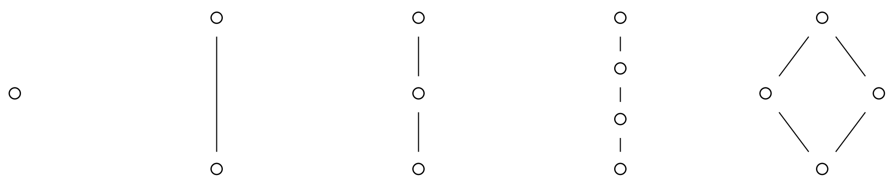


Figure 1.1: The Hasse diagrams of the five lattices with $1 \leq n \leq 4$ elements

A *chain* is a lattice whose induced partial order is a total order. Clearly, in a chain L, for any two elements $a, b \in L$, it holds that either $a \wedge b = a$ and $a \vee b = b$ or that $a \wedge b = b$ and $a \vee b = a$; conversely, if a, b are incomparable, then both $a \wedge b$ and $a \vee b$ are distinct from both a and b. It is obvious that all finite chains of any fixed cardinality are isomorphic (both as lattices and as orders); their Hasse diagrams are like the leftmost four diagrams in Figure 1.1.

We shall refer to the two-element chain ($L = \{\mathbf{0}, \mathbf{1}\}, \mathbf{0} < \mathbf{1}$) as $\mathbb{2}$; it is the only two-element lattice. $\mathbb{2}^n$ shall denote the n-th direct power of $\mathbb{2}$, i.e. the direct product of n copies of $\mathbb{2}$.

1.2 Special classes of lattices

Modular lattices

A lattice $(L, \wedge, \vee)$ is called *modular* if it satisfies the identities

- $\forall\, a, b, c \in L$: $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee (a \wedge c))$; (M1)
- $\forall\, a, b, c \in L$: $(a \vee b) \wedge (a \vee c) = a \vee (b \wedge (a \vee c))$. (M2)

In all lattices, (M1) holds with inequality $\leq$ and (M2) with $\geq$ ([4], p. 71), thus modular lattices are characterized by the fact that the reverse inequalities also hold.

Modularity is a self-dual concept. In fact, only one of the two modular laws is sufficient to characterize modularity, since they are both equivalent to the (self-dual) condition

$$\forall\, a, b, c \in L\colon\; c \leq a \;\Rightarrow\; (a \wedge b) \vee c = a \wedge (b \vee c) \tag{ML}$$

called the *modular law*. The equivalence is proved in [4], pp. 72–73. If $=$ is replaced by $\leq$ in (ML), then the resulting statement, called the *modular inequality*, holds in any lattice.

Modular lattices are a proper subclass of lattices, since the *pentagon* (Figure 1.2) is a lattice which is not modular. Conversely, a lattice is modular iff it does not contain the pentagon as a sublattice: in a nonmodular lattice, pick elements a, b, c such that $c \leq a$ and $(a \wedge b) \vee c < a \wedge (b \vee c)$; then the elements

$$b,\; a \wedge b,\; b \vee c,\; (a \wedge b) \vee c,\; a \wedge (b \vee c)$$

are all distinct and form a pentagon (cf. [4], pp. 109–110). We show first that neither $c \leq b$ nor $b \leq a$ can hold: for, if $c \leq b$, then also $c \leq (a \wedge b)$ by our assumption that $c \leq a$, and therefore $(a \wedge b) \vee c = a \wedge b = a \wedge (b \vee c)$, in contradiction to our assumption that $(a \wedge b) \vee c \neq a \wedge (b \vee c)$. Similarly, if $b \leq a$, then also $(b \vee c) \leq a$, and in this case $a \wedge (b \vee c) = b \vee c = (a \wedge b) \vee c$, so the contradictory equality follows again.

Now we can prove that $a \wedge b, (a \wedge b) \vee c, a \wedge (b \vee c), b \vee c$ form a four-element chain. Clearly non-strict inequalities all hold. No two elements can be equal, because

- $a \wedge b = (a \wedge b) \vee c \;\Rightarrow\; c \leq (a \wedge b) \;\Rightarrow\; c \leq b$;
- $(a \wedge b) \vee c \neq a \wedge (b \vee c)$ by assumption;
- $a \wedge (b \vee c) = b \vee c \;\Rightarrow\; (b \vee c) \leq a \;\Rightarrow\; b \leq a$.

Also, $a \wedge b, b, b \vee c$ clearly form a three-element chain. The last piece of the proof is to show that b is incomparable to both $a \wedge (b \vee c)$ and $(a \wedge b) \vee c$. To this end, note that if two elements are comparable, then their join is equal to one of them and so is their meet. Consider

$$\big((a \wedge b) \vee c\big) \vee b = (a \wedge b) \vee (c \vee b) = b \vee c,$$

which is unequal to both b and $(a \wedge b) \vee c$, and

$$b \wedge \big(a \wedge (b \vee c)\big) = (b \wedge a) \wedge (b \vee c) = b \wedge a,$$

unequal to both b and $a \wedge (b \vee c)$.

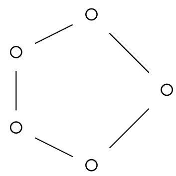


Figure 1.2: The pentagon, characterizing modularity

Distributive lattices

A lattice $(L, \wedge, \vee)$ is called *distributive* if it satisfies the *distributive laws*

- $\forall\, a, b, c \in L\colon\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c);$ (D1)
- $\forall\, a, b, c \in L\colon\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$ (D2)

In other words, we require that $\wedge$ be distributive over $\vee$ and vice versa. Naturally, this concept is self-dual. As with the modular law, inequality versions of these identities hold in any lattice: the former with $\geq$ and the latter with $\leq$ ([4], p. 71).

The two distributive laws are equivalent to each other and to the self-dual condition

$$\forall\, a, b, c \in L\colon\ (a \vee b) \wedge c \leq a \vee (b \wedge c). \tag{DL}$$

A proof can be found in [4], p. 72.

All distributive lattices are modular, as can be seen by a direct computation:

$$\begin{aligned}
&a \wedge (b \vee (a \wedge c)) \\
&= a \wedge \big((b \vee a) \wedge (b \vee c)\big) && \text{(distributivity)}\\
&= (a \wedge (b \vee a)) \wedge (b \vee c) && \text{(associativity)}\\
&= a \wedge (b \vee c) && \text{(absorption)}\\
&= (a \wedge b) \vee (a \wedge c). && \text{(distributivity)}
\end{aligned}$$

The converse is not true: distributive lattices form a proper subclass of modular lattices. The *diamond* (Figure 1.3) plays a similar role for distributive lattices as the pentagon for modular lattices: a modular lattice is distributive iff it does not contain the diamond as a sublattice. Evidently, the diamond is not distributive; for a proof of the converse, take a modular nondistributive lattice with $a \wedge (b \vee c) > (a \wedge b) \vee (a \wedge c)$, then the following elements form a diamond (cf. [4], p. 85 and pp. 109–111):

$$\begin{gathered}
(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) =: u,\\
(a \vee b) \wedge (b \vee c) \wedge (c \vee a) =: v,\\
(a \wedge v) \vee u, \qquad (b \wedge v) \vee u, \qquad (c \wedge v) \vee u.
\end{gathered}$$

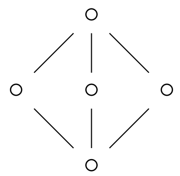


Figure 1.3: The diamond, characterizing distributivity

Complete lattices

For lattices seen as orders, the condition that every set of the form $\{a, b\}$ with $a, b \in L$ (that is, all one- and two-element sets) must have an infimum and a supremum is equivalent to the much more natural condition that any finite nonempty set must have an infimum and a supremum. One direction is trivial; the other one is due to induction on a kind of associative property of infima and suprema: for all $a, b, c \in L$,

$$\inf\{\inf\{a, b\}, c\} = \inf\{a, b, c\} = \inf\{a, \inf\{b, c\}\},$$

and similarly for sup (see [4], pp. 9–10). Because $\wedge$ is also associative, the infimum of every finite nonempty set coincides with its meet, defined inductively:

$$\bigwedge\{x_0\} = x_0; \qquad \bigwedge\{x_0, x_1, \ldots, x_n\} = \bigwedge\{x_0, x_1, \ldots, x_{n-1}\} \wedge x_n.$$

The join of a finite nonempty set is defined similarly, and is equal to the supremum.

A lattice in which *every* subset has an infimum and a supremum is called a *complete lattice*. The concept of completeness is self-dual. From now on we shall write $\bigwedge A$ for $\inf A$ and $\bigvee A$ for $\sup A$ for arbitrary subsets A of a lattice L, and use these as definitions of meets and joins over arbitrary subsets of a complete lattice.

The empty set satisfies $\bigwedge \varnothing = \bigvee L$ and $\bigvee \varnothing = \bigwedge L$, if they exist: $\bigwedge \varnothing$ is an element x satisfying $x \leq y$ for all $y \in \varnothing$ and $x \geq t$ for all $t \in L$ satisfying $t \leq y$ for all $y \in \varnothing$; because of the vacuous universal quantifiers, this simplifies to $x \geq t$ for all $t \in L$, which means $x = \bigvee L$. This implies every finite lattice is complete.

Bounded lattices

An element $\mathbf{0}$ of a lattice $(L, \wedge, \vee)$ is called a *zero* if $\mathbf{0} \leq a$ holds for all $a \in L$. Similarly, an element $\mathbf{1}$ is called a *unit* if $a \leq \mathbf{1}$ holds for all $a \in L$. By antisymmetry, zeros and units are unique if they exist. If the zero and unit are equal in a lattice, then by antisymmetry, the lattice has precisely the one element which is both the zero and the unit. Since the inequalities defining $\mathbf{0}$ and $\mathbf{1}$ imply

$$a \wedge \mathbf{0} = \mathbf{0}, \quad a \vee \mathbf{0} = a, \quad a \wedge \mathbf{1} = a, \quad a \vee \mathbf{1} = \mathbf{1}$$

for all $a \in L$, this makes $\mathbf{0}$ the absorbing element of $\wedge$ and the neutral element of $\vee$, and $\mathbf{1}$ the absorbing element of $\vee$ and the neutral element of $\wedge$. Since $\mathbf{0} \leq a$ holds for all $a \in L$, the conditions $\mathbf{0} = a$ and $\mathbf{0} \geq a$ are equivalent for any $a \in L$; likewise, $\mathbf{1} = a$ and $\mathbf{1} \leq a$ are equivalent.

Clearly, lattices may have "zero divisors" for the meet operation: there may exist nonzero elements a, b such that $a \wedge b = \mathbf{0}$, as in the rightmost lattice of Figure 1.1. On the other hand, for the *join* operation, not only do lattices lack "zero divisors", but an even stronger condition holds: if two elements a and b of a lattice with a zero satisfy $a \vee b = \mathbf{0}$, then it follows that *both* $a = \mathbf{0}$ and $b = \mathbf{0}$. For a proof, simply consider that $\mathbf{0} = a \vee b \geq a$, so $a \leq \mathbf{0}$, which is equivalent to $a = \mathbf{0}$; by symmetry, $b = \mathbf{0}$. Dually, if $a \wedge b = \mathbf{1}$, then $a = \mathbf{1}$ and $b = \mathbf{1}$.

A lattice which has both a zero and a unit is called a *bounded lattice*. Evidently, $\mathbf{0}$ and $\mathbf{1}$ are dual to each other, and boundedness is a self-dual concept. Every complete lattice L is bounded, since the zero and unit are given by $\mathbf{0} = \bigwedge L = \bigvee \varnothing$ and $\mathbf{1} = \bigvee L = \bigwedge \varnothing$ respectively. In particular, every finite lattice is bounded. The family of clopen sets of any topological space, equipped with the operations $\cap$ and $\cup$, forms a bounded lattice which in general is not complete.

In a bounded lattice, an *atom* is an element that covers $\mathbf{0}$; a *co-atom* is covered by $\mathbf{1}$. A *complement* of an element x is an element $\tilde{x}$ such that $x \wedge \tilde{x} = \mathbf{0}$ and $x \vee \tilde{x} = \mathbf{1}$. A complement of an element may not exist, as is the case for the middle element of the three-element chain; or a single element may have multiple complements, as in the diamond, where every non-zero, non-unit element has two complements.

Bounded lattices can also be seen as universal algebras $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ with two additional nullary operations $\mathbf{0}$ and $\mathbf{1}$ that return the zero and the unit, respectively. By such a definition, the class of bounded lattices forms a variety.

1.3 Ortholattices

An *ortholattice* is a bounded lattice $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ with an additional unary operation $'$, called *orthocomplementation*, which satisfies the axioms

- $\forall\, a \in L\colon\ a \wedge a' = \mathbf{0};$ (O1)
- $\forall\, a \in L\colon\ a \vee a' = \mathbf{1};$ (O2)
- $\forall\, a \in L\colon\ a'' = a;$ (O3)
- $\forall\, a, b \in L\colon\ a \leq b \ \Rightarrow\ a' \geq b'.$ (O4)

(O1) and (O2) imply that a' is a complement of a, so every element of an ortholattice has at least one complement. (O4) shows that $\mathbf{0}' = \mathbf{1}$ and $\mathbf{1}' = \mathbf{0}$. (O3) shows that $'$ is a bijection; also, it turns the implication in (O4) into an equivalence and makes the operation $'$ self-dual. This equivalence implies that for any subset A of L with $A' := \{a' \mid a \in A\}$, $\bigwedge A' = \inf A'$ exists iff $\bigvee A = \sup A$ exists and they are equal; likewise, $\bigvee A' = \sup A'$ exists iff $\bigwedge A = \inf A$ exists and they are equal ([6], p. 17). Thus, (O4) implies the *De Morgan laws*: for all $a, b \in L$,

$$(a \wedge b)' = a' \vee b' \quad \text{and} \quad (a \vee b)' = a' \wedge b'. \tag{DML}$$

Conversely, the De Morgan laws imply (O4), since $a \leq b$ implies $a \vee b = b$ and therefore $b' = (a \vee b)' = a' \wedge b' \leq a'$ ([1], pp. 30–31). Thus, the De Morgan laws also imply each other, and ortholattices can be characterized through equations only, therefore the class of ortholattices is a variety. (O4) is known as the *antitone property*; it extends also to the strict order $<$ and the covering relation $\prec$, and it implies that $'$ maps atoms to co-atoms and vice versa.

In an ortholattice L, the relation $\{(a, b) \in L^2 \mid a = b \text{ or } a = b'\}$ is an equivalence relation. Clearly its equivalence classes have at most two elements. If for some element $a \in L$ the equivalence class has only one element, i.e. it holds that $a = a'$, then $\mathbf{0} = a \wedge a' = a \wedge a = a = a' = a' \vee a' = a \vee a' = \mathbf{1}$, so the entire lattice has only one element. Therefore, every finite ortholattice has either one element or else an even number of elements. The mapping $a \mapsto a'$ is an isomorphism of the ortholattice $(L, \wedge, \vee, \mathbf{0}, \mathbf{1}, ')$ onto the dual lattice $(L, \vee, \wedge, \mathbf{1}, \mathbf{0}, ')$, so every ortholattice is isomorphic to its dual. This is not the case for general bounded lattices, as the pair of bounded lattices in Figure 1.4 demonstrates.

Figure 1.4: A pair of mutually dual, non-isomorphic bounded lattices

If $a \leq a'$ holds in an ortholattice for some element a, then $a = a \wedge a' = \mathbf{0}$; so in particular $\exists a\colon\ a \prec a'$ is true only in $\mathbb{2}$. Thus the only ortholattices in which $a' = b$ does not imply $a \neq b$ and $a \not\prec b$ are the one- and two-element ortholattices. Accordingly, we shall represent finite ortholattices with more than two elements through extended Hasse diagrams in which a solid line represents covering and a dashed line connects pairs of non-zero, non-unit orthocomplements, as in Figure 1.5, representing the ortholattice $\mathbb{2}^2$. In larger Hasse diagrams, we shall omit indicating orthocomplementation when it can be uniquely deduced from the axioms (O1)–(O4).

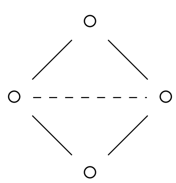


Figure 1.5: $\mathbb{2}^2$, the smallest ortholattice with more than two elements

A distributive ortholattice is called a *Boolean algebra*. The well-known Stone's representation theorem for Boolean algebras states that, for all Boolean algebras B, there exists a set S such that B can be embedded in the Boolean algebra $(\mathcal{P}(S), \cap, \cup, \varnothing, S, X \mapsto S \setminus X)$ via an injective homomorphism, and that for finite

Boolean algebras this homomorphism is bijective. Thus in particular every Boolean algebra is isomorphic to a subalgebra of $\mathbb{2}^I$ for some index set I, and every finite Boolean algebra is isomorphic to $\mathbb{2}^n$ for some $n \in \mathbb{N}$. In particular, the cardinality of every finite Boolean algebra is a power of two. Thus Boolean algebras are the most well-behaved ortholattices. This is elucidated by the fact that $\mathbb{2}$ must be a member of every nontrivial variety of ortholattices, and $\mathbb{2}$ generates all the Boolean algebras through subalgebras and direct products, so the class of Boolean algebras covers the trivial class in the lattice of varieties of ortholattices ([6], pp. 121–123).

If an ortholattice is a Boolean algebra, then x' is the only complement of any element x, i.e. if some element $\tilde{x}$ satisfies $x \wedge \tilde{x} = \mathbf{0}$ and $x \vee \tilde{x} = \mathbf{1}$, then $\tilde{x} = x'$. For a proof, let $\tilde{x}$ be a complement of x in a Boolean algebra, then $x' = x' \wedge \mathbf{1} = x' \wedge (x \vee \tilde{x})$, which by the distributive laws equals $(x' \wedge x) \vee (x' \wedge \tilde{x}) = \mathbf{0} \vee (x' \wedge \tilde{x}) = x' \wedge \tilde{x}$, thus overall $x' = x' \wedge \tilde{x}$, which means $x' \leq \tilde{x}$; with the dual argument, one shows $x' \geq \tilde{x}$ and therefore $x' = \tilde{x}$ by antisymmetry ([4], p. 97).

The same proper inclusion chain "lattice $\supset$ modular lattice $\supset$ distributive lattice" also holds for ortholattices: not all ortholattices are modular, and not all modular ortholattices are distributive. An example of a nondistributive modular ortholattice is the lattice MO_2 or *Chinese lantern* of Figure 1.6, and an example of a nonmodular ortholattice is the lattice O_6 or *benzene ring* of Figure 1.7.

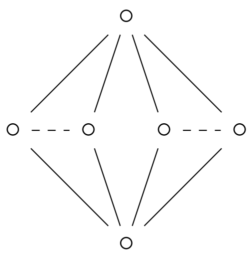


Figure 1.6: MO_2, a nondistributive modular ortholattice

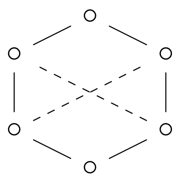


Figure 1.7: O_6, a nonmodular ortholattice

1.4 Orthomodular lattices

There is yet another variety of ortholattices which is contained strictly between the class of all ortholattices and the class of modular ortholattices, namely the *orthomodular lattices.* We shall see that while O_6 does not characterize modularity for ortholattices, it does characterize the weaker condition of orthomodularity. On the other hand, MO_2 plays an important role for subalgebras of orthomodular lattices generated by two elements.

An *orthomodular lattice* is an ortholattice which additionally satisfies the identities

$$\forall\, a, b \in L\colon\; a \wedge (a' \vee (a \wedge b)) = a \wedge b, \quad a \vee (a' \wedge (a \vee b)) = a \vee b. \tag{OM}$$

As with modularity and distributivity, these mutually dual identities imply each other; thus, only one is necessary and orthomodularity is self-dual. However, unlike with modularity and distributivity, the proof is even simpler in this case, since every ortholattice is isomorphic to its dual via orthocomplementation.

The two identities are both equivalent to the pair of mutually dual conditions

$$\forall\, a, b \in L\colon\; \begin{cases} a \leq b & \Rightarrow\; a \vee (a' \wedge b) = b, \\ a \geq b & \Rightarrow\; a \wedge (a' \vee b) = b \end{cases} \tag{OML}$$

of which the version with $\leq$ is called the *orthomodular law.* To prove that (OM) implies (OML), note simply that $a \leq b$ implies $b = a \vee b$ and thus $a \vee (a' \wedge b) = a \vee \big(a' \wedge (a \vee b)\big) = a \vee b = b$. The converse implication follows from simply noticing that for any two $a, b \in L$, we have $a \leq a \vee b$.

Another common characterization of orthomodularity ([6], pp. 22–23) is either of the dual conditions

$$\forall\, a, b \in L\colon\; \begin{cases} (a \leq b \;\text{ and }\; b \wedge a' = \mathbf{0}) & \Rightarrow\; a = b, \\ (a \geq b \;\text{ and }\; b \vee a' = \mathbf{1}) & \Rightarrow\; a = b. \end{cases} \tag{OML*}$$

To prove (OML) implies (OML*), let $a, b \in L$ be such that $a \leq b$ and $b \wedge a' = \mathbf{0}$; then on the one hand $a \vee (a' \wedge b) = b$ by (OML), but on the other hand $a \vee (a' \wedge b) = a \vee \mathbf{0} = a$, thus $a = b$. Conversely, let $a \leq b$; then, since also $a' \wedge b \leq b$ holds, we have $a \vee (a' \wedge b) \leq b$; moreover,

$$b \wedge \big(a \vee (a' \wedge b)\big)' = b \wedge \big(a' \wedge (a' \wedge b)\big)' = (a' \wedge b) \wedge (a' \wedge b)' = \mathbf{0},$$

thus, by (OML*), $a \vee (a' \wedge b) = b$, proving (OML).

Every modular ortholattice is an orthomodular lattice, since by the modular law, $a \leq b$ implies $(b \wedge a') \vee a = b \wedge (a' \vee a) = b \wedge \mathbf{1} = b$. However, as mentioned above, an orthomodular lattice is not the same thing as a modular ortholattice: orthomodularity is a strictly weaker condition than modularity. Figure 2.1, given in [1], p. 42, is an example of an orthomodular lattice which is not modular, as it contains the pentagon as a sublattice.

The benzene ring characterizes orthomodularity among ortholattices. Clearly it is not orthomodular; conversely, suppose L is not orthomodular, then there are elements s, t such that $s \leq t$, $t \wedge s' = \mathbf{0}$ and yet $s \neq t$. Then the elements $\mathbf{0}, s, s', t, t', \mathbf{1}$ form a benzene ring ([1], p. 54). Firstly $s \leq t$ and $s \neq t$ imply $s < t$; by (O4), $s' \geq t'$, and because $'$ is bijective, $s' \neq t'$; thus $s' > t'$. Thus we have the inequality chains

$$\mathbf{0} \leq s < t \leq \mathbf{1}; \qquad \mathbf{0} \leq t' < s' \leq \mathbf{1}.$$

We first show that all inequalities here are strict: if $\mathbf{0} = s$, equivalently $s' = \mathbf{1}$, then on the one hand $t \wedge s' = \mathbf{0}$ by assumption, but on the other hand $t \wedge s' = t \wedge \mathbf{1} = t$, thus $s = \mathbf{0} = t$, in contradiction to $s \neq t$. Likewise, if $\mathbf{0} = t'$, or $t = \mathbf{1}$, then $\mathbf{0} = t \wedge s' = \mathbf{1} \wedge s' = s'$, contradicting $s' \neq t'$.

Finally, we need to show none of the following pairs are comparable:

$$(s, s'), \quad (s, t'), \quad (t, s'), \quad (t, t').$$

If any element u was comparable with its complement, then either $u = \mathbf{0}$ or $u = \mathbf{1}$, which we have ruled out for all of s, t, s', t'. If t and s' were comparable, then s and t' would also be comparable, and vice versa; in both cases $t \wedge s'$ would have to be equal to either t or s', but by assumption it is equal to $\mathbf{0}$. Thus $\mathbf{0}, s, s', t, t', \mathbf{1}$ form a benzene ring, as desired.

Chapter 2

Structure of Orthomodular Lattices

2.1 Commutativity

The relation x *commutes with* y, written xCy, for two elements x and y of an ortholattice L is defined as

$$xCy \quad \text{iff} \quad x = (x \wedge y) \vee (x \wedge y').$$

We shall write $xC'y$ and read x *dually commutes with* y for the dual relation, obtained by exchanging $\wedge$ and $\vee$. Similarly, $x\tilde{C}y$ shall denote the converse relation, i.e. yCx. An element c is called *central* in L if xCc holds for all $x \in L$.

We begin with a few elementary remarks. The relations C and C' are not empty, as for any element $x \in L$ it holds that $\mathbf{0}Cx$, $\mathbf{1}Cx$, $xC\mathbf{0}$, and $xC\mathbf{1}$, and all of the above with C' in place of C. The relation C is a superset of $\leq$, since for any two elements $x, y \in L$, $x \leq y$ implies $(x \wedge y) \vee (x \wedge y') = x \vee (x \wedge y') = x$ by absorption. Similarly, C' is a superset of $\geq$. In particular, both C and C' are reflexive. Moreover, for any two elements $x, y \in L$, xCy trivially implies xCy', and likewise for C'. In particular, every element commutes and dually commutes with its complement. Finally, the notation C' for the dual of C is motivated by the equivalence $xC'y \Leftrightarrow x'Cy'$ for all $x, y \in L$, which follows from the De Morgan laws.

However, C is generally not symmetric nor transitive, and C and C' are in general different relations. It is a defining characteristic of *orthomodular* lattices in particular that C agrees with C' and both are symmetric.

Theorem 2.1.1 ([1], pp. 45–46)**.** *In any ortholattice L the following conditions are all equivalent:*

(i) L is orthomodular;

(ii) $C = \tilde{C}'$;

(iii) $C = C'$;

(iv) $C = \tilde{C}$.

Proof.

- (i) $\Rightarrow$ (ii): Assume xCy, so $x = (x \wedge y) \vee (x \wedge y')$. By the absorption law, $x \vee y = y \vee (y \wedge x) \vee (y' \wedge x) = y \vee (y' \wedge x)$. Now define

$$a := y; \qquad b := (y \vee x) \wedge (y \vee x').$$

It suffices to show that a and b are equal, as this will prove xCy implies $yC'x$; the converse implication is equivalent. To this end we note first that

$$a \wedge b = y \wedge (y \vee x) \wedge (y \vee x') = y \wedge (y \vee x') = y = a$$

by applying the absorption law twice, so that $a \leq b$; and secondly,

$$\begin{aligned} b \wedge a' &= (y \vee x) \wedge (y \vee x') \wedge y' \\ &= (y \vee x) \wedge (y' \wedge x)' \wedge y' \\ &= (y \vee x) \wedge \big((y' \wedge x) \vee y\big)' \\ &= (x \vee y) \wedge (x \vee y)' \\ &= \mathbf{0}. \end{aligned}$$

Now the conditions of (OML*) are met and it follows that $a = b$.

- (ii) $\Rightarrow$ (iii): By assumption and the elementary remarks we obtain

$$xCy \;\Rightarrow\; xCy' \;\Rightarrow\; y'C'x \;\Rightarrow\; y'C'x' \;\overset{\text{(ii)}}{\Longrightarrow}\; x'Cy' \;\Rightarrow\; xC'y.$$

- (i) $\Rightarrow$ (iv): From what was proven so far, orthomodularity implies $C = \tilde{C}'$ and $C = C'$. Then

$$xCy \;\Rightarrow\; yC'x \;\Rightarrow\; yCx.$$

- (iii) $\Rightarrow$ (i): Suppose $C = C'$ and take two elements $a, b \in L$ such that $a \leq b$. Then $bC'a$, which by assumption implies bCa, which together with $a \leq b$ gives $b = (b \wedge a) \vee (b \wedge a') = a \vee (a' \wedge b)$. By (OML), L is orthomodular.

- (iv) $\Rightarrow$ (i): Suppose $C = \tilde{C}$ and take $a \leq b$, which implies aCb. By assumption, bCa follows, and this along with $a \leq b$ implies orthomodularity just like in the point above.

□

Corollary 2.1.2. *In an orthomodular lattice L and $x, y \in L$, for all*

$$\mathbf{x}, \hat{\mathbf{x}} \in \{x, x'\}, \quad \mathbf{y}, \hat{\mathbf{y}} \in \{y, y'\}, \quad \mathbf{C}, \hat{\mathbf{C}} \in \{C, C', \tilde{C}, \tilde{C}'\},$$

$\mathbf{xCy}$ *is equivalent to* $\hat{\mathbf{x}}\hat{\mathbf{C}}\hat{\mathbf{y}}$. □

A crucial property of commutativity in orthomodular lattices is given by the *Foulis–Holland theorem*: commutativity between triplets of elements implies their distributivity.

Theorem 2.1.3 (weak Foulis–Holland; [1], pp. 47–49)**.** *Let L be an orthomodular lattice and $a, b, c \in L$. If one of a, b, c commutes with the other two (e.g. aCb and aCc), then any permutation of a, b, c satisfies the distributive laws.*

Proof. Let aCb and aCc. We need to verify twelve distributive identities: two for each of the six permutations of a, b, c. However, we can reduce the work. Define

$$\begin{aligned} D_\wedge(a,b,c) &\Leftrightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c); \\ D_\vee(a,b,c) &\Leftrightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \end{aligned}$$

Then, in any distributive identity involving a, b, c, we can always exchange the rightmost and middle elements by the fact that $\wedge$ and $\vee$ are commutative, and we can always exchange b and c by exchanging their roles, since the assumptions are symmetric in b and c. Overall, this lets us reduce the six permutations to just two: specifically, for $\Diamond \in \{\wedge, \vee\}$,

$$D_\Diamond(a,b,c) \Leftrightarrow D_\Diamond(a,c,b); \quad D_\Diamond(b,c,a) \Leftrightarrow D_\Diamond(b,a,c) \Leftrightarrow D_\Diamond(c,a,b) \Leftrightarrow D_\Diamond(c,b,a),$$

reducing the number of identities from twelve to four. Moreover, the assumption aCb and aCc also implies $a'Cb'$ and $a'Cc'$ by Theorem 2.1.1; thus, defining $\wedge' = \vee$ and $\vee' = \wedge$,

$$D_\Diamond(a', b', c') \Leftrightarrow D_{\Diamond'}(a,b,c),$$

so that the dual identities follow by considering a', b', c'. This leaves only two identities to prove (e.g. $D_\wedge(a,b,c)$ and $D_\wedge(b,a,c)$), which we now proceed to do.

(i) $D_\wedge(a,b,c)$: Define

$$x := (a \wedge b) \vee (a \wedge c) \quad \text{and} \quad y := a \wedge (b \vee c).$$

In any lattice $x \leq y$ holds (cf. [4], p. 71). We will show that $y \wedge x' = \mathbf{0}$, so that we may apply (OML*) to obtain $x = y$.
By assumption, aCb and therefore $b'C'a$ by Theorem 2.1.1. Therefore

$$a \wedge b' = a \wedge (b' \vee a) \wedge (b' \vee a') = a \wedge (b' \vee a')$$

by absorption. Similarly, aCc, so that $a \wedge c' = a \wedge (c' \vee a')$. Now

$$\begin{aligned} x' \wedge y &= \big((a \wedge b) \vee (a \wedge c)\big)' \wedge a \wedge (b \vee c) \\ &= (a \wedge b)' \wedge (a \wedge c)' \wedge a \wedge (b \vee c) \\ &= (a' \vee b') \wedge \underbrace{(a' \vee c') \wedge a}_{=a \wedge c'} \wedge (b \vee c) \\ &= \underbrace{(a' \vee b') \wedge a}_{=a \wedge b'} \wedge c' \wedge (b \vee c) \\ &= a \wedge b' \wedge c' \wedge (b \vee c) \\ &= a \wedge (b \vee c)' \wedge (b \vee c) \\ &= a \wedge \mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

(ii) $D_\wedge(b,a,c)$: By a similar argument, defining this time

$$x := (b \wedge a) \vee (b \wedge c) \quad \text{and} \quad y := b \wedge (a \vee c),$$

using aCb to conclude $a'C'b$ and therefore $(a' \vee b') \wedge b = a' \wedge b$, and using aCc to conclude $cC'a$ and therefore $a' \wedge (a \vee c) = a' \wedge c$, one obtains using the same sequence of steps as above that $x' \wedge y = \mathbf{0}$.

□

An important consequence of the Foulis–Holland theorem is that in orthomodular lattices, C is compatible not only with $'$, but also with $\wedge$ and $\vee$:

Theorem 2.1.4 ([1], pp. 49–50)**.** *Let L be an orthomodular lattice and $a, b, c \in L$. If aCb and aCc, then also $aC(b \wedge c)$ and $aC(b \vee c)$.*

Proof. Observe that if one of the assertions is proved, the other one follows, since for $\Diamond \in \{\wedge, \vee\}$, assuming ($aCb$ and aCc) $\Rightarrow aC(b \Diamond c)$ is proved, we have

$$(aCb \text{ and } aCc) \;\Rightarrow\; (aCb' \text{ and } aCc') \;\Rightarrow\; aC(b' \Diamond c') \;\Rightarrow\; aC(b \Diamond' c)' \;\Rightarrow\; aC(b \Diamond' c)$$

using the notation from the proof of Theorem 2.1.3. It therefore suffices to show one of the two assertions; we will show $aC(b \vee c)$. Assume therefore aCb and aCc, then the condition of Theorem 2.1.3 is satisfied, so a, b, c are distributive; moreover, since also $a'Cb$ and $a'Cc$, we get that a', b, c are distributive; finally, we also have bCa and cCa. Putting all that together, we obtain

$$\begin{aligned}
&\big((b \vee c) \wedge a\big) \vee \big((b \vee c) \wedge a'\big) \\
&= (b \wedge a) \vee (c \wedge a\,) \vee (b \wedge a') \vee (c \wedge a') \\
&= (b \wedge a) \vee (b \wedge a') \vee (c \wedge a\,) \vee (c \wedge a') \\
&= b \vee c,
\end{aligned}$$

thus $(b \vee c)Ca$ and therefore $aC(b \vee c)$. □

By an inductive argument and by applying $C = \tilde{C}$ we come to the following corollary:

Corollary 2.1.5 ([1], p. 51)**.** *In an orthomodular lattice, if for some natural numbers n, m all elements a_i for all $i \in \{0, 1, \dots, n-1\}$ commute with all elements b_j for all $j \in \{0, 1, \dots, m-1\}$, then all elements of the form*

$$p(a_0, a_1, \dots, a_{n-1}), \qquad q(b_0, b_1, \dots, b_{m-1})$$

also commute, where p and q are n-ary and m-ary functions defined by terms in n and m variables and the operations $\wedge$, $\vee$, and $'$. □

The significance of the above corollary is that elements of the form $p(x_0, x_1, \dots, x_{n-1})$ for some natural number n and some n-ary function p defined by a term in $\wedge$, $\vee$, and $'$ are precisely the elements of a subalgebra generated by the elements $x_0, x_1, \dots, x_{n-1}$. Therefore, if we can find a generating set for a subalgebra of an orthomodular lattice in which every element commutes with every other, then every element of the whole subalgebra commutes with every other.

2.2 Characterization of Boolean algebras

We know that an ortholattice is a Boolean algebra iff it is distributive. The following Theorem 2.2.1 allows for an alternative characterization of Boolean algebras among ortholattices through the commutativity relation.

Theorem 2.2.1 ([1], p. 65)**.** *In any ortholattice L the statements (i)-(iv) are all equivalent:*

(i) $C = L^2$*;*

(ii) C *is an equivalence relation;*

(iii) C *is transitive;*

(iv) L *is a Boolean algebra.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial. For (iii) $\Rightarrow$ (i), note merely that $aC\mathbf{0}$ and $\mathbf{0}Cb$ hold for any a, b, so by transitivity aCb.

For (iv) $\Rightarrow$ (i), assume L is distributive, then for any two elements a, b we have

$$a = a \wedge \mathbf{1} = a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b'),$$

thus aCb, proving $C = L^2$.

For (i) $\Rightarrow$ (iv), suppose that $C = L^2$; then in particular C is symmetric, so L is orthomodular and we can apply Theorem 2.1.3; since for any three elements a, b, c we have aCb and aCc, all distributive identities between a, b, c hold and L is distributive. ☐

Another characterization of Boolean algebras is through uniqueness of complements. While general bounded lattices may not have complements for every element, and ortholattices only ensure the existence of at least one complement of every element x (namely x'), Boolean algebras always have *exactly* the one complement x' for every element x. What's interesting is that the converse also holds; and it allows to single out Boolean algebras from all ortholattices. We first need an auxiliary result.

Theorem 2.2.2 ([6], pp. 25–26)**.** *In an ortholattice L the complements of an element x are all of the form*

$$\big(y \wedge (y' \vee x')\big) \vee (y' \wedge x'),$$

where $y \in L$. If L is orthomodular, then the converse also holds, i.e. for every $y \in L$, elements of this form are complements of x.

Proof. We show first that all complements of x are of this form. Let $\tilde{x}$ be a complement of x, then by the De Morgan laws $\tilde{x}'$ is a complement of x' and thus

$$\big(\tilde{x} \wedge (\tilde{x}' \vee x')\big) \vee (\tilde{x}' \wedge x') = (\tilde{x} \wedge \mathbf{1}) \vee \mathbf{0} = \tilde{x},$$

thus $\tilde{x}$ is of the required form with $y = \tilde{x}$.

We now show that if L is orthomodular, then all elements of this form are complements of x. First we prove that, for all y, the meet of $\big(y \wedge (y' \vee x')\big) \vee (y' \wedge x')$ with x is $\mathbf{0}$.

We apply a technique known as *focusing*: in an expression of the form $(a \wedge b) \vee c$ or $(a \vee b) \wedge c$, we identify one of a, b, c which commutes with the other two, so that we may apply distributivity by Theorem 2.1.3. In our case, in $\big(\big(y \wedge (y' \vee x')\big) \vee (\underline{y' \wedge x'})\big) \wedge x$, $y' \wedge x'$ commutes with both x and $y \wedge (y' \vee x')$, thus we can apply distributivity:

$$
\begin{aligned}
&\Big(\big(y \wedge (y' \vee x')\big) \vee (y' \wedge x')\Big) \wedge x \\
&= \big(\big(y \wedge (y' \vee x')\big) \wedge x\big) \vee \big((y' \wedge x') \wedge x\big) \\
&= \big(\underbrace{(y \wedge x) \wedge (y \wedge x)'}_{=\mathbf{0}}\big) \vee \big(y' \wedge \underbrace{(x' \wedge x)}_{=\mathbf{0}}\big) \\
&= \mathbf{0} \vee (y' \wedge \mathbf{0}) \\
&= \mathbf{0} \vee \mathbf{0} \\
&= \mathbf{0}.
\end{aligned}
$$

Next we show that the join is $\mathbf{1}$. Here we focus on $y' \vee x'$, which commutes with both y and x:

$$
\begin{aligned}
&\Big(\big(y \wedge (y' \vee x')\big) \vee (y' \wedge x')\Big) \vee x \\
&= \Big(\big(y \wedge (\underline{y' \vee x'})\big) \vee x\Big) \vee (y' \wedge x') \\
&= \Big((y \vee x) \wedge \underbrace{\big((y' \vee x') \vee x\big)}_{=\mathbf{1}}\Big) \vee (y' \wedge x') \\
&= \big((y \vee x) \wedge \mathbf{1}\big) \vee (y' \wedge x') \\
&= (y \vee x) \vee (y \vee x)' \\
&= \mathbf{1}.
\end{aligned}
$$

□

We can now prove that Boolean lattices are exactly the uniquely complemented ortholattices:

Theorem 2.2.3 ([6], p. 26)**.** *An ortholattice L is Boolean iff every element x has exactly one complement, namely x'.*

Proof.

- (⇒): This has been proved already in a remark in Chapter 1. For an alternative proof using Theorem 2.2.2, note that every complement of x is of the form

$\big(y \wedge (y' \vee x')\big) \vee (y' \wedge x')$ and then the distributive laws imply

$$\begin{aligned}
&\big(y \wedge (y' \vee x')\big) \vee (y' \wedge x') \\
&= \big(\underbrace{(y \wedge y')}_{=\mathbf{0}} \vee (y \wedge x')\big) \vee (y' \wedge x') \\
&= (y \wedge x') \vee (y' \wedge x') \\
&= \underbrace{(y \vee y')}_{=\mathbf{1}} \wedge x' \\
&= x'.
\end{aligned}$$

- ($\Leftarrow$): Suppose now that for all $x \in L$, x' is the only complement of x.

  We first show that L is orthomodular. Let $x, y \in L$ be such that $x \leq y$ and $y \wedge x' = \mathbf{0}$; we will show $x = y$. Note that $x \leq y$ implies $x = x \wedge y$ and therefore $x \wedge y' = (x \wedge y) \wedge y' = x \wedge (y \wedge y') = x \wedge \mathbf{0} = \mathbf{0}$; on the other hand, $y \wedge x' = \mathbf{0}$ implies, by taking complements, that $x \vee y' = \mathbf{1}$. Thus y' is a complement of x, and so, by the hypothesis of unique complements, $y' = x'$ and thus $y = x$.

  Now that it has been established that L is orthomodular, we can apply Theorems 2.1.3, 2.2.1, and 2.2.2. We know all complements of x are precisely the elements of the form $\big(y \wedge (y' \vee x')\big) \vee (y' \wedge x')$ for arbitrary y; thus, by the hypothesis of unique complements,

$$\big(y \wedge (y' \vee x')\big) \vee (y' \wedge x') = x'$$

  holds for any $x, y \in L$. Let therefore $x, y \in L$ be arbitrary; we will prove yCx, from which it will follow that $C = L^2$ and therefore that L is Boolean.

  First, take the meet with y on both sides:

$$y \wedge \Big(\big(y \wedge (y' \vee x')\big) \vee (y' \wedge x')\Big) = y \wedge x'.$$

  Now we focus on y, which commutes with both $y \wedge (y' \vee x')$ and with $(y \vee x)' = y' \wedge x'$:

$$\begin{aligned}
&\underline{y} \wedge \Big(\big(y \wedge (y' \vee x')\big) \vee (y' \wedge x')\Big) \\
&= \big(\underbrace{y \wedge y}_{=y} \wedge (y' \vee x')\big) \vee (\underbrace{y \wedge y'}_{=\mathbf{0}} \wedge x') \\
&= \big(y \wedge (y' \vee x')\big) \vee (\mathbf{0} \wedge x') \\
&= \big(y \wedge (y' \vee x')\big) \vee \mathbf{0} \\
&= y \wedge (y' \vee x').
\end{aligned}$$

  Thus we have $y \wedge (y' \vee x') = y \wedge x'$. Now take the join with $y \wedge x$ on both sides:

$$\big(y \wedge (y' \vee x')\big) \vee (y \wedge x) = (y \wedge x') \vee (y \wedge x).$$

Now if we can show that the term on the left-hand side is equal to y, then we will have proven yCx. We use the orthomodular law:

$$a \leq b \;\Rightarrow\; a \vee (a' \wedge b) = b.$$

Define $a := y \wedge x$ and $b := y$; then $a \leq b$ is satisfied, and so

$$\big(y \wedge (y' \vee x')\big) \vee (y \wedge x) = (b \wedge a') \vee a = b.$$

Therefore we conclude $y = (y \wedge x) \vee (y \wedge x')$, thus yCx.

□

Remark: Another relation which can be seen as a measure of distributivity in ortholattices is the relation of *perspectivity*, defined as follows: In a bounded lattice L with $a, b \in L$, a is *perspective* to b (written $a \sim b$) if a and b have a common complement, i.e. there exists $c \in L$ such that $a \wedge c = \mathbf{0} = b \wedge c$ and $a \vee c = \mathbf{1} = b \vee c$. In light of the above Theorem 2.2.3, we have the following corollary:

Corollary 2.2.4 ([6], p. 73)**.** *An ortholattice is Boolean iff $\sim$ is the identity relation, i.e. $(\sim) = \{(x, x) \mid x \in L\}$.* □

2.3 Intervals

For a lattice L and $a, b \in L, a \leq b$, the *interval* $[a, b]$ is the subset $\{x \in L \mid a \leq x \leq b\}$. (Unlike its notation may suggest, an interval of L need not be a chain if L is not one.) By the properties of the induced partial order and by its transitivity, for all $x, y \in L$ we have

$$a \leq x \leq b \text{ and } a \leq y \leq b \;\Rightarrow\; a \leq (x \wedge y) \leq b \text{ and } a \leq (x \vee y) \leq b,$$

thus $[a, b]$ is a sublattice of L. Note that, of course, intervals of ortholattices are generally *not* closed under orthocomplementation: in any non-trivial ortholattice, the interval $[\mathbf{0}, \mathbf{0}] = \{\mathbf{0}\}$ does not contain $\mathbf{0}' = \mathbf{1}$. Intervals of ortholattices can even have a structure where *no* unary operation could be an orthocomplementation: take for instance the three-element interval $[\mathbf{0}, t] = \{\mathbf{0}, s, t\}$ of the benzene ring.

However, under additional assumptions, one *can* define a new orthocomplementation on intervals of L. Let $a, b \in L$, $a \leq b$ and define the operations

$$x^{a\uparrow b} := (a \vee x') \wedge b; \qquad x^{a\downarrow b} := \; a \vee (x' \wedge b).$$

The interval $[a, b]$ is closed under both: for all $x \in [a, b]$, $x^{a\uparrow b} \leq b$ because $x^{a\uparrow b}$ is a meet with b, and $x^{a\uparrow b} \geq a$ because it is equal to

$$x^{a\uparrow b} = (a \vee x') \wedge b = (a \vee x') \wedge (a \vee b),$$

the meet of two elements which are both joins with a. The proof for $x^{a\downarrow b}$ is analogous. The next theorem states that if these two operations are equal on $[a,b]$, and under additional conditions of commutativity, they define an orthocomplementation on $[a,b]$ that makes it an ortholattice.

Theorem 2.3.1 (cf. [1], pp. 55–57)**.** *Let $(L,\wedge,\vee,\mathbf{0},\mathbf{1},{}')$ be an ortholattice and $a,b\in L$ with $a\le b$. If for all $x\in[a,b]$, $x^{a\uparrow b}=x^{a\downarrow b}$ (we shall write $x^{a|b}$ for both), and x both commutes and dually commutes with both a and b, then the algebra $([a,b],\wedge,\vee,a,b,{}^{a|b})$ is an ortholattice. In particular, both additional conditions are fulfilled if L is orthomodular, in which case $([a,b],\wedge,\vee,a,b,{}^{a|b})$ is also orthomodular.*

Proof.

- (O1): Because $x\in[a,b]$,

  $$x\wedge x^{a|b}=x\wedge((a\vee x')\wedge b)=(x\wedge b)\wedge(x'\vee a)=x\wedge(x'\vee a).$$

  Because both x and $x^{a|b}$ are in $[a,b]$, and intervals are closed under joins and meets, $x\wedge(x'\vee a)$ is also in $[a,b]$, thus it commutes with a:

  $$\begin{aligned}&x\wedge(x'\vee a)\\&=\big(x\wedge(x'\vee a)\wedge a\big)\vee\big(x\wedge(x'\vee a)\wedge a'\big)\\&=\big((x\wedge a)\wedge(a\vee x')\big)\vee\big((x\wedge a')\wedge(x'\vee a)\big)\\&=\big(\underbrace{a\wedge(a\vee x')}_{=a}\big)\vee\big(\underbrace{(x'\vee a)'\wedge(x'\vee a)}_{=\mathbf{0}}\big)\\&=a\vee\mathbf{0}\\&=a.\end{aligned}$$

  Thus $x\wedge x^{a|b}=a$, proving (O1).

- (O2): Similarly,

  $$x\vee x^{a|b}=x\vee(a\vee(x'\wedge b))=(x\vee a)\vee(x'\wedge b)=x\vee(x'\wedge b).$$

  A similar argument as above, considering that $x\vee(x'\wedge b)$ dually commutes with b, shows that it is equal to b, thus $x\vee x^{a|b}=b$, proving (O2).

- (O3): Note that, because all $x\in[a,b]$ commute with a,

  $$a\vee(a'\wedge x)=(a\wedge x)\vee(a'\wedge x)=(x\wedge a)\vee(x\wedge a')=x.$$

  Similarly, since all x dually commute with b,

  $$b\wedge(b'\vee x)=(b\vee x)\wedge(b'\vee x)=(x\vee b)\wedge(x'\vee b)=x.$$

  Therefore,

  $$\begin{aligned}(x^{a|b})^{a|b}&=\Big(a\vee\big((a\vee x')\wedge b\big)'\Big)\wedge b\\&=\big(a\vee(a\vee x')'\vee b'\big)\wedge b\\&=\big(a\vee(a'\wedge x)\vee b'\big)\wedge b\\&=(x\vee b')\wedge b\\&=x.\end{aligned}$$

- (O4): If $x \leq y$, then

$$\begin{aligned} x' &\geq y'; \\ x' \wedge b &\geq y' \wedge b; \\ a \vee (x' \wedge b) &\geq a \vee (y' \wedge b) \end{aligned}$$

  by the isotone property of the lattice order. Thus $x^{a|b} \geq y^{a|b}$.

- Orthomodularity of L implies the conditions of the theorem:
  $a \leq x \leq b$ implies aCx and xCb, which by orthomodularity implies all of xCa, $xC'a$, xCb, $xC'b$, which is the second condition. Also, $a \leq x \leq b$ implies that a commutes with both x' and b, thus from distributivity we conclude

$$x^{a\uparrow b} = (a \vee x') \wedge b = (a \vee x') \wedge (a \vee b) = a \vee (x' \wedge b) = x^{a\downarrow b}.$$

- Orthomodularity of L implies orthomodularity of $[a, b]$:
  We show (OML). For $x, y \in [a, b]$, assume $x \leq y$. Then xCy and thus yCx'. Also, because $a \leq y$, we have aCy and yCa; finally, by Theorem 2.1.4, $yC(a \vee x')$. We focus on y:

$$\begin{aligned} & x \vee (x^{a|b} \wedge y) \\ &= x \vee \big(\big((a \vee x') \wedge b\big) \wedge y\big) \\ &= x \vee \big((a \vee x') \wedge b \wedge y\big) \\ &= x \vee \big((a \vee x') \wedge \underline{y}\big) \\ &= \big(\underbrace{x \vee (a \vee x')}_{=\mathbf{1}}\big) \wedge (x \vee y) \\ &= x \vee y \\ &= y. \end{aligned}$$

□

Two maps of a lattice L into the interval $[a, b]$ ($a, b \in L,\ a \leq b$) are the *upper contraction* $a{\uparrow}b$ and the *lower contraction* $a{\downarrow}b$, defined by

$$\begin{aligned} a{\uparrow}b(x) &:= (a \vee x) \wedge b; \\ a{\downarrow}b(x) &:= \ a \vee (x \wedge b). \end{aligned}$$

By the modular inequality, for all $a, b \in L$ with $a \leq b$, and all $x \in L$, $a{\downarrow}b(x) \leq a{\uparrow}b(x)$, which justifies the names (cf. [1], p. 58). The condition that they be equal for all $a, b \in L$ with $a \leq b$ and all $x \in L$ is exactly equivalent to the modularity of L. Evidently, both $a{\uparrow}b$ and $a{\downarrow}b$ are the identity on $[a, b]$, hence they are idempotent and therefore surjective; also, it is clear that for all $x \in L$,

$$a{\uparrow}b(x') = x^{a\uparrow b} \quad \text{and} \quad a{\downarrow}b(x') = x^{a\downarrow b}.$$

Recall that an element c of an ortholattice L is called *central* if xCc holds for all $x \in L$. Clearly, if c is central, so is c'. Because $'$ is a bijection from L to L, a central element equivalently satisfies $x'Cc$ for all $x \in L$, which in turn is equivalent to $xC'c$. Thus, every element of L both commutes and dually commutes with a central element.

The theorem below states that if L is an ortholattice, under additional conditions similar to – but stronger than – those in Theorem 2.3.1, the upper and lower contractions are homomorphisms of L onto the ortholattice $[a, b]$.

Theorem 2.3.2 (cf. [6], pp. 20–21)**.** *Let $(L, \wedge, \vee, \mathbf{0}, \mathbf{1}, ')$ be an ortholattice and $a, b \in L$ with $a \leq b$. If $a{\uparrow}b = a{\downarrow}b$ on all of L (we shall write $a|b$ for both), and both a and b are central in L, then $a|b$ is a homomorphism of $(L, \wedge, \vee, \mathbf{0}, \mathbf{1}, ')$ onto the ortholattice $([a, b], \wedge, \vee, a, b, {}^{a|b})$.*

Proof.

- Preserving $\mathbf{0}$ and $\mathbf{1}$: This is trivial, since $a|b(\mathbf{0}) = a \vee (\mathbf{0} \wedge b) = a \vee \mathbf{0} = a$ and $a|b(\mathbf{1}) = (a \vee \mathbf{1}) \wedge b = \mathbf{1} \wedge b = b$.

- Preserving $'$: We must show $a|b(x') = \big(a|b(x)\big)^{a|b}$. First, note that xCa and $xC'b$ for all $x \in L$ implies

$$\begin{aligned} x \vee a &= \big((x \wedge a) \vee (x \wedge a')\big) \vee a = \big((x \wedge a) \vee a\big) \vee (x \wedge a') = a \vee (a' \wedge x), \\ x \wedge b &= \big((x \vee b) \wedge (x \vee b')\big) \wedge b = \big((x \vee b) \wedge b\big) \wedge (x \vee b') = b \wedge (b' \vee x) \end{aligned}$$

  by absorption. From this we obtain

$$\begin{aligned} \big(a|b(x)\big)^{a|b} &= \big(a \vee (x \wedge b)\big)^{a|b} \\ &= a \vee \Big(\big(a \vee (x \wedge b)\big)' \wedge b\Big) \\ &= a \vee \big(a' \wedge (x \wedge b)' \wedge b\big) \\ &= a \vee \big((x \wedge b)' \wedge b\big) \\ &= a \vee \big((x' \vee b') \wedge b)\big) \\ &= a \vee (x' \wedge b) \\ &= a|b(x'). \end{aligned}$$

- Preserving $\wedge$ and $\vee$: Since the conditions of this theorem imply the conditions of Theorem 2.3.1, it suffices to show that one of the binary operations is preserved, since the other one will then also be preserved by the fact that $'$ is preserved and that the De Morgan laws hold in the ortholattice $[a, b]$. We will show it for $\vee$.

  Firstly, for all $x, y \in L$ we show the implication

$$(x \leq b \text{ and } y \leq b') \;\Rightarrow\; x = (x \vee y) \wedge b.$$

  On the one hand, $x \leq b$ by assumption and $x \leq x \vee y$, thus also $x \leq (x \vee y) \wedge b$. On the other hand, $y \leq b'$ and so $(x \vee y) \wedge b \leq (x \vee b') \wedge b = x \wedge b = x$ by the isotone property, where $(x \vee b') \wedge b = x \wedge b$ was proved in the previous point.

Next, for all $x, y \in L$ we can show that x, y, and b distribute:

$$(x \vee y) \wedge b = (x \wedge b) \vee (y \wedge b).$$

This follows from the above:

$$\begin{aligned}
&(x \vee y) \wedge b \\
&= \Big(\big((x \wedge b) \vee (x \wedge b')\big) \vee \big((y \wedge b) \vee (y \wedge b')\big)\Big) \wedge b \\
&= \Big(\big(\underbrace{(x \wedge b) \vee (y \wedge b)}_{=:\mathbf{x}}\big) \vee \big(\underbrace{(x \wedge b') \vee (y \wedge b')}_{=:\mathbf{y}}\big)\Big) \wedge b.
\end{aligned}$$

We have $\mathbf{x} \leq b$ and $\mathbf{y} \leq b'$, therefore $(\mathbf{x} \vee \mathbf{y}) \wedge b = \mathbf{x} = (x \wedge b) \vee (y \wedge b)$, which completes the proof.

Finally, we can show $a|b(x \vee y) = a|b(x) \vee a|b(y)$:

$$\begin{aligned}
&a|b(x) \vee a|b(y) \\
&= a \vee (x \wedge b) \vee a \vee (y \wedge b) \\
&= (a \vee a) \vee (x \wedge b) \vee (y \wedge b) \\
&= a \vee (x \wedge b) \vee (y \wedge b) \\
&= a \vee \big((x \vee y) \wedge b\big) \\
&= a|b(x \vee y).
\end{aligned}$$

□

Remark: Unlike with Theorem 2.3.1, the orthomodularity of L is clearly not sufficient to guarantee the conditions of Theorem 2.3.2. Theorem 2.2.1 implies that the most general class of ortholattices that satisfies these conditions for every interval is the class of Boolean algebras. In fact, orthomodularity is not even sufficient to guarantee that the lower and upper contractions coincide; that condition is equivalent to *modularity*, which is strictly stronger. The orthomodular, but not modular, lattice in Figure 2.1 gives a counterexample: any two-element interval whose lower end is one of a, b, c (the upper end is one of c', b', a') generates two contractions that map d to different values.

Nevertheless, it is noteworthy that if an interval is of the form $[\mathbf{0}, b]$ or $[a, \mathbf{1}]$, then the conditions are almost fully fulfilled: the two contractions coincide in this case, and $\mathbf{0}$ and $\mathbf{1}$ are always central. In this case it suffices to require that the other endpoint of the interval also be central. This is exactly the situation in the main theorem of this section.

Theorem 2.3.3 ([6], p. 20)**.** *Let $(L, \wedge, \vee, \mathbf{0}, \mathbf{1}, ')$ be an ortholattice and $c \in L$. Then the following statements are equivalent:*

(i) c is central in L;

(ii) $\phi_c : x \mapsto (x \wedge c, x \wedge c')$ is an isomorphism of L onto the ortholattice $[\mathbf{0}, c] \times [\mathbf{0}, c']$, where the ortholattice operations on the intervals are defined as in Theorem 2.3.1; the inverse mapping to ϕ_c is $(x, y) \mapsto x \vee y$.

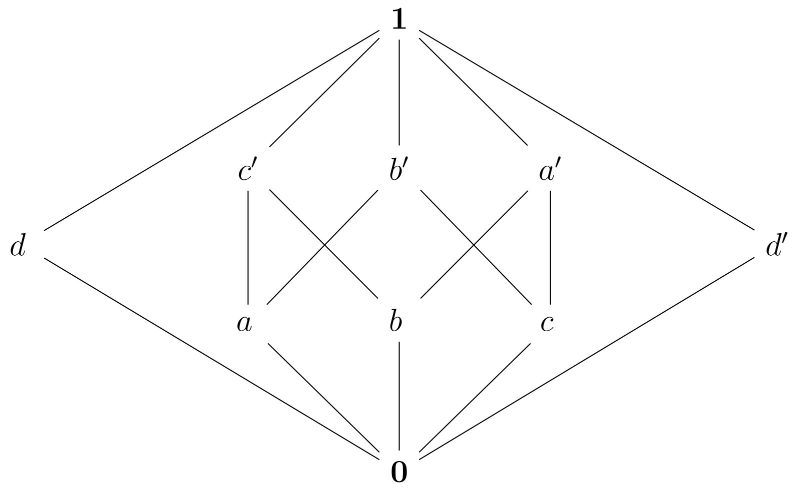


Figure 2.1: An orthomodular lattice which is not modular

Proof.

- (ii) ⇒ (i) is straightforward: Assume the given mapping ϕ_c is an isomorphism. Then we have

$$\begin{aligned}&\phi_c\big((x \wedge c) \vee (x \wedge c')\big)\\ &= \phi_c(x \wedge c) \vee \phi_c(x \wedge c')\\ &= (x \wedge c \wedge c,\ x \wedge c \wedge c') \vee (x \wedge c' \wedge c,\ x \wedge c' \wedge c')\\ &= (x \wedge c,\ \mathbf{0}) \vee (\mathbf{0},\ x \wedge c')\\ &= ((x \wedge c) \vee \mathbf{0},\ (\mathbf{0} \vee (x \wedge c'))\\ &= (x \wedge c, x \wedge c')\\ &= \phi_c(x).\end{aligned}$$

  Because ϕ_c is injective by assumption, $x = (x \wedge c) \vee (x \wedge c')$ for all $x \in L$, so c is central.

- (i) ⇒ (ii): Assume c is central. As outlined in the remark above, the conditions of Theorem 2.3.2 are satisfied for both $[\mathbf{0}, c]$ and $[\mathbf{0}, c']$ (since c and c' are either both central or both not central), and so the contractions

$$\mathbf{0}|c : x \mapsto x \wedge c; \qquad \mathbf{0}|c' : x \mapsto x \wedge c'$$

  are homomorphisms of L onto the ortholattices $[\mathbf{0}, c]$ and $[\mathbf{0}, c']$, whose units are c and c', respectively, and whose orthocomplementations are

$$x^{\mathbf{0}|c} = x' \wedge c; \qquad x^{\mathbf{0}|c'} = x' \wedge c'.$$

  Therefore $\phi_c : x \mapsto (\mathbf{0}|c(x), \mathbf{0}|c'(x)) = (x \wedge c, x \wedge c')$ is a homomorphism into the direct product ortholattice $[\mathbf{0}, c] \times [\mathbf{0}, c']$. It remains to show that this homomorphism is bijective.

  To show that ϕ_c is injective, let $x, y \in L$ with $\phi_c(x) = \phi_c(y)$. Equality of the ordered pairs implies their components are both equal, thus $x \wedge c = y \wedge c$ and

$x \wedge c' = y \wedge c'$. Now because c is central, we have xCc and yCc, thus

$$x = (x \wedge c) \vee (x \wedge c') = (y \wedge c) \vee (y \wedge c') = y.$$

To show ϕ_c is surjective, recall that, as part of the proof of Theorem 2.3.2, we have shown the implication

$$(x \leq c \text{ and } y \leq c') \;\Rightarrow\; x = (x \vee y) \wedge c$$

provided c is central. By symmetry, and by noting that c' is also central, we also have the conclusion $y = (x \vee y) \wedge c'$. Then it follows that ϕ_c is surjective, because for every $x \in [\mathbf{0}, c], y \in [\mathbf{0}, c']$, it holds that (x, y) is the image of $x \vee y$ under ϕ_c. This also shows that $(x, y) \mapsto x \vee y$ is the inverse mapping to ϕ_c.

□

The above Theorem 2.3.3 is of importance in inductive proofs of statements about finite ortholattices. It will also be central to the characterization of the free orthomodular lattice with two generators, as it too will turn out to have nontrivial central elements, and thus is isomorphic to a nontrivial direct product. Finally, an important consequence of Theorem 2.3.3 is the following strengthening of Theorem 2.1.3, stating that if some element a of an orthomodular lattice commutes with two others b and c, then not only do a, b, c satisfy the distributive laws, but any meets and joins of them also do, i.e. the generated *sublattice* is distributive.

The *sublattice generated* by a subset A of a lattice L, for which we will write $[A]$, is the closure of A under $\wedge$ and $\vee$; it is the smallest sublattice of L containing A as a subset. Note that if L is an ortholattice, a sublattice of L need *not* be closed under orthocomplementation. The closure under $\wedge$, $\vee$, *and* the orthocomplementation is the *subalgebra generated* by A, which we will denote by $[\![A]\!]$. It is clear that if ϕ is a lattice homomorphism, then $\phi[A] = \big[\phi(A)\big]$; similarly, if ϕ also preserves the orthocomplementation between ortholattices, then $\phi[\![A]\!] = \big[\!\big[\phi(A)\big]\!\big]$.

Theorem 2.3.4 (strong Foulis–Holland; [6], p. 25)**.** *Let L be an orthomodular lattice and $a, b, c \in L$. If one of a, b, c commutes with the other two (e.g. aCb and aCc), then the sublattice $\big[\{a, b, c\}\big]$ is distributive.*

Proof. Consider $\tilde{L} := \big[\!\big[\{a, b, c\}\big]\!\big]$. Being a subalgebra of L, $\tilde{L}$ is also an orthomodular lattice. In $\tilde{L}$, a commutes with both b and c, and it also commutes with itself; because of Theorem 2.1.4, a commutes with every term function of a, b, c in $\wedge$, $\vee$, and $'$, that is, every element of $\tilde{L}$. Thus a is central in $\tilde{L}$, and by Theorem 2.3.3, $\tilde{L}$ is isomorphic to $[\mathbf{0}, a] \times [\mathbf{0}, a']$.

Now consider the following sublattices of L:

- The sublattice generated by $\{\mathbf{1}, b, c\}$ in L is $L_a := \{\mathbf{1}, b, c, b \wedge c, b \vee c\}$. It is easy to see that this lattice is distributive: e.g. it is isomorphic to the first lattice of Figure 1.4, which has no pentagon and no diamond.
- The sublattice generated by $\{\mathbf{0}, b, c\}$ in L is $L_{a'} := \{\mathbf{0}, b, c, b \wedge c, b \vee c\}$, which is isomorphic to the second lattice of Figure 1.4 and is likewise distributive.

Next, note the following:

- $\mathbf{0}|a(\{a,b,c\}) = \{a\wedge a, a\wedge b, a\wedge c\} = \{a, a\wedge b, a\wedge c\} = \mathbf{0}|a(\{\mathbf{1},b,c\})$. The sublattice generated by $\{\mathbf{1},b,c\}$ is L_a, which is distributive; thus also $\mathbf{0}|a(L_a) = \mathbf{0}|a[\{\mathbf{1},b,c\}] = \big[\mathbf{0}|a(\{\mathbf{1},b,c\})\big] = \big[\mathbf{0}|a(\{a,b,c\})\big] = \mathbf{0}|a[\{a,b,c\}]$ is distributive.
- $\mathbf{0}|a'(\{a,b,c\}) = \{a'\wedge a, a'\wedge b, a'\wedge c\} = \{\mathbf{0}, a'\wedge b, a'\wedge c\} = \mathbf{0}|a'(\{\mathbf{0},b,c\})$. Likewise, the sublattice generated by $\{\mathbf{0},b,c\}$ is $L_{a'}$, which is distributive, so $\mathbf{0}|a'(L_{a'}) = \mathbf{0}|a'[\{a,b,c\}]$ is distributive.

Finally, using the notation of Theorem 2.3.3, $\phi_a[\{a,b,c\}]$ is distributive, being the direct product of the two distributive lattices $\mathbf{0}|a[\{a,b,c\}]$ and $\mathbf{0}|a'[\{a,b,c\}]$. However, since ϕ_a is an isomorphism, it follows that also $[\{a,b,c\}]$ is a distributive sublattice of $\tilde{L}$ and thus of L. □

We finish this section with a few closing remarks on the strong Foulis–Holland theorem. The proof presented here relies on the small size of lattices generated by two elements x, y: in the worst case, when x and y are incomparable, the (free) lattice generated by x, y still has only four elements and is isomorphic to the rightmost lattice of Figure 1.1; it is distributive. This technique does not extend to ortholattices, which have significantly more elements in the general case. For this reason, this proof cannot be extended to a stronger assertion about the generated *subalgebra* rather than the sublattice.

In fact, such a stronger assertion is not even true: if elements a, b, c of an orthomodular lattice L have the property that aCb and aCc, then $[\![\{a,b,c\}]\!]$ is in general *not* a Boolean algebra. The orthomodular lattice depicted in Figure 2.2, given in [6], pp. 31–32, provides a counterexample: call its atoms b, s, a', t, c from left to right, then we have aCb and aCc and the lattice equals $[\![\{a,b,c\}]\!]$, but, containing the diamond as a sublattice, it is not distributive. If L is an orthomodular lattice and $a, b, c \in L$ have the property that *every* one of them commutes with the other two, then clearly all of $\{a,b,c\}$ commute with each other, and so, by Corollary 2.1.5, every element of $[\![\{a,b,c\}]\!]$ commutes with every other; by Theorem 2.2.1, this means $[\![\{a,b,c\}]\!]$ is distributive. However, under just the assumptions of Theorem 2.3.4, namely that only *one* element must commute with the other two – it could be that aCb and aCc, but *not* necessarily bCc – we can in general only conclude that $[\![\{a,b,c\}]\!]$ is *modular* (see [6], p. 27), not distributive.

If, however, we consider only *two* elements, then the corresponding assertion (commonly used as an equivalent definition of commutativity in orthomodular lattices) *does* hold, and is an easy consequence of Corollary 2.1.5 and Theorem 2.2.1:

Corollary 2.3.5 (cf. [6], pp. 22–23)**.** *Let L be an orthomodular lattice and $a, b \in L$. Then the following statements are equivalent:*

(i) aCb*;*

(ii) $[\![\{a,b\}]\!]$ *is distributive.*

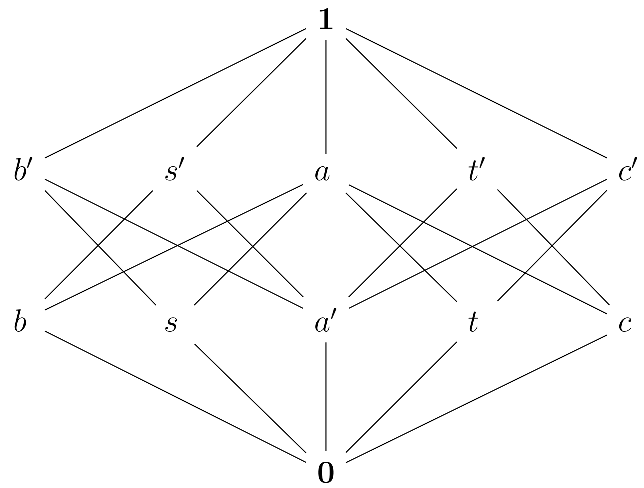


Figure 2.2: A nondistributive OML generated by $\{a, b, c\}$ with aCb and aCc

Proof. If $[\![\{a, b\}]\!]$ is distributive, i.e. a Boolean algebra, then by Theorem 2.2.1, all of its elements commute, in particular a commutes with b, proving (ii) $\Rightarrow$ (i). Conversely, if aCb, then by orthomodularity, bCa and so all elements of $\{a, b\}$ commute with one another; thus, by Corollary 2.1.5, every element of $[\![\{a, b\}]\!]$ commutes with every other and $[\![\{a, b\}]\!]$ is distributive, proving (i) $\Rightarrow$ (ii). $\square$

Chapter 3

Free Orthomodular Lattice with Two Generators

The idea of a *free algebra* over a given class $\mathbf{K}$ of algebras of a given type is a central concept in universal algebra. Informally, a free algebra is "the most general algebra" of a given type generated by a given number of elements. Formally, let τ be a type, $\mathbf{K}$ a class of algebras of type τ, $F \in \mathbf{K}$, and $X \subseteq F$ (X is called the *generating set*); then F is called *free over X in* $\mathbf{K}$ if

(i) X generates F (i.e. F is the closure of X under the algebraic operations);

(ii) for all algebras A in $\mathbf{K}$ and all maps $\phi : X \to A$ there exists a homomorphism $\overline{\phi} : F \to A$ extending ϕ, i.e. satisfying $\overline{\phi}\big|_X = \phi$. If such a homomorphism exists, it is unique ([4], p. 77; [5], p. 67).

The existence of free algebras is clear in the most common cases: if $\mathbf{K}$ is a variety, then for any generating set X, an algebra free over X in $\mathbf{K}$ always exists ([4], p. 80). Uniqueness is likewise guaranteed: if both F and G are free algebras over the same set X in the same class $\mathbf{K}$, then there exists an isomorphism $\phi : F \to G$ such that $\phi\big|_X = \mathrm{id}_X$; in other words, free algebras are unique up to isomorphism if they exist ([4], p. 77, [5], p. 67). Accordingly, we will write $F_{\mathbf{K}}(\kappa)$ for the free algebra over the variety $\mathbf{K}$ with a generating set of cardinality κ; if the generating set is finite, we will also write $F_{\mathbf{K}}(\alpha, \beta, \ldots, \omega)$ for $F_{\mathbf{K}}(\{\alpha, \beta, \ldots, \omega\})$.

For a class of algebras $\mathbf{K}$ and $n \in \mathbb{N}$, $F_{\mathbf{K}}(n)$ is of interest because every algebra A in $\mathbf{K}$ generated by n or fewer elements is a homomorphic image, and thus a quotient algebra, of $F_{\mathbf{K}}(n)$ (a homomorphism being given by the unique extension of any map from the generators of $F_{\mathbf{K}}(n)$ to the generators of A, cf. [5], p. 67). Because homomorphic images preserve identities, any identity that holds in $F_{\mathbf{K}}(n)$ also holds in any algebra in $\mathbf{K}$ generated by n elements; in particular, since any identity in n variables holds in an algebra iff it holds in all its n-generated subalgebras, every identity in n variables that holds in $F_{\mathbf{K}}(n)$ holds in every algebra of $\mathbf{K}$. Therefore, if $F_{\mathbf{K}}(n)$ is finite for some n, the validity of identities in n variables in all of $\mathbf{K}$ may be verified simply by computation in the finite free algebra.

The aim of this chapter is to fully characterize the free orthomodular lattice with two generators, $F_{\mathbf{OML}}(\alpha, \beta)$. We shall see that this orthomodular lattice is isomorphic to the direct product of $\mathbb{2}^4$, the two-generated free Boolean algebra, with MO_2; in particular, it is finite, having 96 elements. We will also present a method outlined in [7] to represent the elements of $F_{\mathbf{OML}}(\alpha, \beta)$ in a standard form that allows for simpler arithmetic, similar to what is possible with the disjunctive or conjunctive normal forms in a Boolean algebra.

3.1 Structure of the Two-Generated Free OML

Note that if an orthomodular lattice L is generated by two elements α and β, then $\alpha \wedge \beta$ commutes with both α and β, and therefore, by Corollary 2.1.5, with every element of L. The same can also be said of elements such as $\alpha \vee \beta$, $\alpha' \wedge \beta$, $\alpha \vee \beta'$, etc. All of these are central in L, and Theorem 2.3.3 then implies that L is decomposable into a direct product via any of these central elements.

For $F_{\mathbf{OML}}(\alpha, \beta)$, one of these central elements (more precisely, one pair of central elements which are orthocomplements) will be of particular importance, as the components of the corresponding direct product will have particularly simple forms. These elements are the *lower commutator* $\alpha \Downarrow \beta$ and *upper commutator* $\alpha \Uparrow \beta$, defined as

$$\alpha \Downarrow \beta := (\alpha \wedge \beta) \vee (\alpha \wedge \beta') \vee (\alpha' \wedge \beta) \vee (\alpha' \wedge \beta');$$
$$\alpha \Uparrow \beta := (\alpha \vee \beta) \wedge (\alpha \vee \beta') \wedge (\alpha' \vee \beta) \wedge (\alpha' \vee \beta').$$

These are dual to each other; by the De Morgan laws, they are also clearly complements of one another. Their names (given in [1], p. 86) are by analogy to the lower and upper contraction: the lower commutator, like the lower contraction, is a join of meets; the upper commutator, like the upper contraction, is a meet of joins. Both commutators are central, which is a straightforward application of Corollary 2.1.5: they are a join and meet, respectively, of central elements. Moreover, the name "commutator" for both of them stems from the fact that they characterize commutativity in an orthomodular lattice:

Theorem 3.1.1 ([1], pp. 86–87; [6], pp. 26–27)**.** *In an orthomodular lattice L the following conditions are equivalent for any $a, b \in L$:*

(i) aCb;

(ii) $a \Downarrow b = \mathbf{1}$;

(iii) $a \Uparrow b = \mathbf{0}$.

Proof.

- (ii) and (iii) are obviously equivalent, since the lower and upper commutator are complements.

- (i) ⇒ (ii): Assume aCb, then by orthomodularity $a'Cb$ and therefore
$$a \Downarrow b = \underbrace{(a \wedge b) \vee (a \wedge b')}_{=a} \vee \underbrace{(a' \wedge b) \vee (a' \wedge b')}_{=a'} = a \vee a' = \mathbf{1}.$$

- (iii) ⇒ (i): Define
$$x := a; \qquad y := (a \vee b) \wedge (a \vee b').$$
We will verify the conditions of (OML*). First, notice that $x \leq y$ always holds, because of $a \leq a \vee b$ and $a \leq a \vee b'$. For the same reason, $a' \leq (a' \vee b) \wedge (a' \vee b')$. Therefore, by the isotone property,
$$\begin{aligned} a' &\leq (a' \vee b) \wedge (a' \vee b') \\ a' \wedge (a \vee b) \wedge (a \vee b') &\leq (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b') \\ a' \wedge (a \vee b) \wedge (a \vee b') &\leq a \Uparrow b. \end{aligned}$$
By assumption, $a \Uparrow b = \mathbf{0}$ and so $a' \wedge (a \vee b) \wedge (a \vee b') = y \wedge x' \leq \mathbf{0}$, meaning $y \wedge x' = \mathbf{0}$. Therefore, by (OML*), $x = y$, which means $aC'b$. By orthomodularity, we conclude aCb.

□

In what follows, L is an orthomodular lattice generated by two elements α, β (not necessarily free). We define $c := \alpha \Downarrow \beta$ and $c' := \alpha \Uparrow \beta$ and we define ortholattice operations on the intervals $[\mathbf{0}, c]$ and $[\mathbf{0}, c']$ as in Theorem 2.3.1. By Theorem 2.3.3, L is isomorphic to $[\mathbf{0}, c] \times [\mathbf{0}, c']$ with these operations, and we will now analyze the two components of this direct product representation more closely. We first examine $[\mathbf{0}, c']$. From here onwards, the six elements of MO_2 (Figure 1.6) will be denoted in fixed-width type: $\mathtt{0}, \mathtt{a}, \mathtt{A}, \mathtt{b}, \mathtt{B}, \mathtt{1}$, where $\mathtt{0}$ is the zero, $\mathtt{1}$ is the unit, $\mathtt{a}' = \mathtt{A}$, and $\mathtt{b}' = \mathtt{B}$.

Theorem 3.1.2 ([1], pp. 77–78; [6], p. 27)**.** *$[\mathbf{0}, c']$ is a homomorphic image of* MO_2*.*

Proof. Since L is generated by α, β, and the map $x \mapsto x \wedge c'$ defined in Theorem 2.3.3 is a homomorphism from L onto $[\mathbf{0}, c']$, it follows that $[\mathbf{0}, c']$ has generators $\alpha \wedge c', \beta \wedge c'$. By the absorption law,
$$\begin{aligned} \alpha \wedge c' &= \alpha \wedge (\alpha \vee \beta) \wedge (\alpha \vee \beta') \wedge (\alpha' \vee \beta) \wedge (\alpha' \vee \beta') \\ &= \alpha \wedge (\alpha' \vee \beta) \wedge (\alpha' \vee \beta'); \\ \beta \wedge c' &= \beta \wedge (\alpha \vee \beta) \wedge (\alpha' \vee \beta) \wedge (\alpha \vee \beta') \wedge (\alpha' \vee \beta') \\ &= \beta \wedge (\alpha \vee \beta') \wedge (\alpha' \vee \beta'). \end{aligned}$$
Similarly, $\alpha' \wedge c' = \alpha' \wedge (\alpha \vee \beta) \wedge (\alpha \vee \beta')$ and $\beta' \wedge c' = \beta' \wedge (\alpha \vee \beta) \wedge (\alpha' \vee \beta)$. We examine the joins and meets of $M := \{\alpha \wedge c', \alpha' \wedge c', \beta \wedge c', \beta' \wedge c'\}$:

- Since c' is central in L, by Theorem 2.1.3 we have $(x \wedge c') \vee (y \wedge c') = (x \vee y) \wedge c'$ for all $x, y \in L$. But notice that, for all $\boldsymbol{\alpha} \in \{\alpha, \alpha'\}$ and $\boldsymbol{\beta} \in \{\beta, \beta'\}$, $\boldsymbol{\alpha} \vee \boldsymbol{\beta}$ is already one of the four terms of which c' is the meet, thus it vanishes by the idempotent law when taking the meet with c', yielding simply $(\boldsymbol{\alpha} \wedge c') \vee (\boldsymbol{\beta} \wedge c') = c'$. Likewise, by commutativity, $(\alpha \wedge c') \vee (\alpha' \wedge c') = c'$ and $(\beta \wedge c') \vee (\beta' \wedge c') = c'$. Thus we see that the join of any two distinct elements of M is c'.

- By the idempotent law, we have $(x \wedge c') \wedge (y \wedge c') = (x \wedge y) \wedge c'$ for all $x, y \in L$. For all $\boldsymbol{\alpha} \in \{\alpha, \alpha'\}$ and $\boldsymbol{\beta} \in \{\beta, \beta'\}$, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ is a complement of one of the four terms in c'; thus, when taking the meet with c', it collapses the expression into $\mathbf{0}$. Therefore the meet of any two distinct elements of M is $\mathbf{0}$.

By the homomorphism property of the map $x \mapsto x \wedge c'$, we know that $(\alpha \wedge c',\ \alpha' \wedge c')$, $(\beta \wedge c',\ \beta' \wedge c')$, $(\mathbf{0}, c')$ are pairs of orthocomplements in $[\mathbf{0}, c']$; moreover, $\mathbf{0}$ is the zero and c' is the unit. It follows that $[\mathbf{0}, c'] = \{\mathbf{0}, \alpha \wedge c', \alpha' \wedge c', \beta \wedge c', \beta' \wedge c', c'\}$; its elements behave just like the six elements of MO_2 under the algebraic operations. Thus $[\mathbf{0}, c']$ is a homomorphic image of MO_2. □

Theorem 3.1.3. MO_2 *is a simple lattice; any homomorphic image of* MO_2 *is either isomorphic to* MO_2 *or it is the one-element lattice.*

Proof. Let $\mathtt{0}, \mathtt{a}, \mathtt{A}, \mathtt{b}, \mathtt{B}, \mathtt{1}$ be the elements of MO_2. We will show the statement via congruence relations: namely, if any two distinct elements of MO_2 are identified, then all its elements are identified. Let therefore $\approx$ be a congruence relation on MO_2.

- If $\mathtt{0} \approx \mathtt{1}$, then every element x is identified with $\mathtt{0}$, since $\mathtt{0} = \mathtt{0} \wedge x \approx \mathtt{1} \wedge x = x$; therefore all elements are identified.
- If some element x is identified with a complement y such that $x \wedge y = \mathtt{0}$ and $x \vee y = \mathtt{1}$, then

  $$\mathtt{0} = x \wedge y \ \approx\ x \wedge x = x \ \approx\ y = y \vee y \ \approx\ x \vee y = \mathtt{1},$$

  thus $\mathtt{0} \approx \mathtt{1}$.
- If a non-zero, non-unit element (WLOG, $\mathtt{a}$) is identified with $\mathtt{0}$, then

  $$\mathtt{b} = \mathtt{b} \vee \mathtt{0} \ \approx\ \mathtt{b} \vee \mathtt{a} = \mathtt{1} = \mathtt{B} \vee \mathtt{a} \ \approx\ \mathtt{B} \vee \mathtt{0} = \mathtt{B},$$

  thus $\mathtt{b} \approx \mathtt{B}$.
- If a non-zero, non-unit element is identified with $\mathtt{1}$, then its orthocomplement (which is also non-zero, non-unit) is identified with $\mathtt{0}$.

Because out of all non-zero, non-unit elements of MO_2, any pair of distinct elements is a pair of complements, this covers all cases of distinct elements. Thus any congruence relation on MO_2 identifies either only equal elements or else all elements, so MO_2 is simple. □

Corollary 3.1.4. *In* $F_{\mathbf{OML}}(\alpha, \beta)$*,* $[\mathbf{0}, c']$ *is isomorphic to* MO_2*.*

Proof. Were $[0, c']$ instead isomorphic to the one-element lattice, then $\mathbf{0}$ would have to equal c' in $F_{\mathbf{OML}}(\alpha, \beta)$, which would imply that the identity

$$a \Uparrow b = \mathbf{0}$$

would hold in every orthomodular lattice L. However, by Theorem 3.1.1, this implies aCb, whence by Theorem 2.2.1, all orthomodular lattices would be Boolean algebras. □

Let L again be an orthomodular lattice generated by α and β. We found that in that case $[\mathbf{0}, c']$ is a homomorphic image of MO_2. Now we turn to the other factor, $[\mathbf{0}, c]$:

Theorem 3.1.5 ([1], p. 78–79; [6], p. 27)**.** *$[\mathbf{0}, c]$ is a Boolean algebra.*

Proof. In analogy to Theorem 3.1.2, we find that $\alpha \wedge c, \beta \wedge c$ are the generators of $[\mathbf{0}, c]$. Applying distributivity by Theorem 2.1.3,

$$\begin{aligned}
\alpha \wedge c &= \alpha \wedge \big((\alpha \wedge \beta) \vee (\alpha \wedge \beta') \vee (\alpha' \wedge \beta) \vee (\alpha' \wedge \beta')\big) \\
&= \big(\alpha \wedge \big((\alpha \wedge \beta) \vee (\alpha \wedge \beta')\big)\big) \vee \big(\alpha \wedge \big((\alpha' \wedge \beta) \vee (\alpha' \wedge \beta')\big)\big) \\
&= (\alpha \wedge \alpha \wedge \beta) \vee (\alpha \wedge \alpha \wedge \beta') \vee (\underbrace{\alpha \wedge \alpha' \wedge \beta}_{=\mathbf{0}}) \vee (\underbrace{\alpha \wedge \alpha' \wedge \beta'}_{=\mathbf{0}}) \\
&= (\alpha \wedge \beta) \vee (\alpha \wedge \beta').
\end{aligned}$$

Analogously, $\beta \wedge c = (\alpha \wedge \beta) \vee (\alpha' \wedge \beta)$. Because $\alpha \wedge \beta$, $\alpha \wedge \beta'$, $\alpha' \wedge \beta$, $\alpha' \wedge \beta'$ are central elements, by Corollary 2.1.5, $\alpha \wedge c$ and $\beta \wedge c$ commute with each other. Therefore, by Corollary 2.3.5, $[\mathbf{0}, c] = [\![\{\alpha \wedge c, \beta \wedge c\}]\!]$ is a Boolean algebra. □

Theorem 3.1.6. *$F_{\mathbf{Bool}}(\alpha, \beta)$, the free Boolean algebra with two generators, is isomorphic to $\mathbb{2}^4$.*

Proof. We will make use of the disjunctive normal form. Consider any element of $F_{\mathbf{Bool}}(\alpha, \beta)$; it is a term t in the variables α, β and the operations $\wedge, \vee, \mathbf{0}, \mathbf{1}, '$. Apply the following transformations to t:

- Eliminate $\mathbf{0}$ and $\mathbf{1}$ by replacing them with $\alpha \wedge \alpha'$ and $\alpha \vee \alpha'$, respectively.
- Apply the De Morgan laws repeatedly on subterms of the form $(x \wedge y)'$ and $(x \vee y)'$ until there are no more subterms of this form; then, apply $x'' = x$ repeatedly until there are no multiple applications of $'$; now, all instances of $'$ in t are applied either to α or to β.
- Apply the distributive law (D1) repeatedly until all meets of joins are replaced by joins of meets. Now t has the form

  $$t(\alpha, \beta) = \bigvee_{i \in I} \bigwedge_{j \in J_i} x_{ij},$$

  where each x_{ij} is one of $\alpha, \beta, \alpha', \beta'$. We define $T_i := \{x_{ij} \mid j \in J_i\}$.
- By the idempotent law, any repeat elements among the x_{ij} for any J_i can be eliminated, therefore

  $$t(\alpha, \beta) = \bigvee_{i \in I} \bigwedge_{j \in J_i} x_{ij} \;=\; \bigvee_{i \in I} \bigwedge T_i.$$

- If some T_i contains both α and α', or both β and β', then $\bigwedge T_i = \mathbf{0}$, so T_i can be removed from t. Now, every T_i contains at most one of α and α' and at most one of β and β'. We consider three cases:

(a) $T_i = \{x, y\}$ with $x \neq y$ (it contains exactly one of α and α' and exactly one of β and β'): Leave it.

(b) $T_i = \{x\}$, WLOG $x = \alpha$: Replace it by $T_{i_0} = \{\alpha, \beta\}$ and $T_{i_1} = \{\alpha, \beta'\}$. The value of t is unchanged, since $(\alpha \wedge \beta) \vee (\alpha \wedge \beta') = \alpha$, since in a Boolean algebra any two elements commute.

(c) $T_i = \varnothing$: Replace it by $T_{i_0} = \{\alpha, \beta\}$, $T_{i_1} = \{\alpha, \beta'\}$, $T_{i_2} = \{\alpha', \beta\}$, and $T_{i_3} = \{\alpha', \beta'\}$. The value of t is unchanged, since

$$(\alpha \wedge \beta) \vee (\alpha \wedge \beta') \vee (\alpha' \wedge \beta) \vee (\alpha' \wedge \beta') = \alpha \Downarrow \beta = \mathbf{1} = \bigwedge \varnothing,$$

since by Theorem 3.1.1 the lower commutator of any two elements is $\mathbf{1}$ in a Boolean algebra.

- In every case we transformed the T_i in such a way that each of them now contains exactly one of α and α' and exactly one of β and β'.
- Define $T := \{T_i \mid i \in I\}$. By the idempotent law, any repeat elements among the T_i can be eliminated, giving t its final form

$$t(\alpha, \beta) = \bigvee_{U \in T} \bigwedge U.$$

The elements of T now all contain exactly one of α and α' and exactly one of β and β'. There are only four such sets: $\{\alpha, \beta\}, \{\alpha, \beta'\}, \{\alpha', \beta\}, \{\alpha', \beta'\}$. Therefore, defining $S := \big\{\{\alpha, \beta\}, \{\alpha, \beta'\}, \{\alpha', \beta\}, \{\alpha', \beta'\}\big\}$, there are only $16 = 2^4 = |\mathcal{P}(S)|$ possibilities for the set T, and we have shown that $F_{\mathbf{Bool}}(\alpha, \beta)$ is a homomorphic image of $(\mathcal{P}(S), \cap, \cup, \varnothing, S, X \mapsto S \setminus X)$, which itself is clearly isomorphic to $\mathbb{2}^4$. To see that $F_{\mathbf{Bool}}(\alpha, \beta)$ has exactly 16 elements, consider that the 16-element Boolean algebra $\mathbb{2}^4$ has the generators $\alpha := (\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0})$ and $\beta := (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1})$, since

$$\begin{aligned}
\alpha \wedge \beta' &= (\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0});\\
\alpha' \wedge \beta' &= (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0});\\
\alpha \wedge \beta &= (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0});\\
\alpha' \wedge \beta &= (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}),
\end{aligned}$$

and every other element of $\mathbb{2}^4$ is a (possibly empty) join of these four, and neither α nor β alone generates all 16 elements of $\mathbb{2}^4$. Since every Boolean algebra with two generators must be the homomorphic image of $F_{\mathbf{Bool}}(\alpha, \beta)$, it follows that $F_{\mathbf{Bool}}(\alpha, \beta)$ must have at least 16 elements; thus, in combination with the above, it has exactly 16 elements and it is isomorphic to $\mathbb{2}^4$. □

At this stage we remark that the proof of Theorem 3.1.6 can be generalized to show that $F_{\mathbf{Bool}}(n)$, the free Boolean algebra with n generators, is isomorphic to $\mathbb{2}^{2^n}$ for all $n \in \mathbb{N}$. In particular, every finitely generated Boolean algebra is finite. The same is not true of finitely generated free orthomodular lattices, as even the free orthomodular lattice with three generators is already infinite (see [1], pp. 89–94).

Corollary 3.1.7. *In $F_{\mathbf{OML}}(\alpha, \beta)$, $[\mathbf{0}, c]$ is isomorphic to $\mathbb{2}^4$.*

Proof. We need to show that in $F_{\mathbf{OML}}(\alpha, \beta)$, $[\mathbf{0}, c]$ is a free Boolean algebra with two generators $\alpha \wedge c, \beta \wedge c$. Since α, β generate $F_{\mathbf{OML}}(\alpha, \beta)$, by the homomorphism property of $x \mapsto x \wedge c$, we have that $\alpha \wedge c, \beta \wedge c$ generate $[\mathbf{0}, c]$. We must show that $[\mathbf{0}, c]$ is free and that $\alpha \wedge c, \beta \wedge c$ are distinct (so that they are in fact *two* generators). To this end, let B be a Boolean algebra, and let ϕ be a map from $\{\alpha \wedge c, \beta \wedge c\}$ into B; we must find a homomorphic extension $\overline{\phi}$ from $[\mathbf{0}, c]$ into B extending ϕ.

Note that B, being a Boolean algebra, is in particular orthomodular. Define the map ψ from $\{\alpha, \beta\}$ into B by

$$\psi(\alpha) = \phi(\alpha \wedge c); \qquad \psi(\beta) = \phi(\beta \wedge c).$$

Then, by the freeness of $F_{\mathbf{OML}}(\alpha, \beta)$, there exists a homomorphic extension $\overline{\psi}$ from $F_{\mathbf{OML}}(\alpha, \beta)$ into B extending ψ. We claim that the map $\overline{\psi}\big|_{[\mathbf{0},c]}$ is the desired homomorphic extension of ϕ from $[\mathbf{0}, c]$ into B, proving that $[\mathbf{0}, c]$ is free. It is a homomorphism, so we only need to show that it extends ϕ:

$$\begin{aligned}
\overline{\psi}(\alpha \wedge c) &= \overline{\psi}\big((\alpha \wedge \beta) \vee (\alpha \wedge \beta')\big) \\
&= \big(\psi(\alpha) \wedge \psi(\beta)\big) \vee \big(\psi(\alpha) \wedge \psi(\beta)'\big) \\
&= \big(\phi(\alpha \wedge c) \wedge \phi(\beta \wedge c)\big) \vee \big(\phi(\alpha \wedge c) \wedge \phi(\beta \wedge c)'\big) \\
&= \phi(\alpha \wedge c),
\end{aligned}$$

where the last equality is because ϕ maps into a Boolean algebra, where all elements commute; the computation for $\beta \wedge c$ is analogous. Thus $[\mathbf{0}, c]$ is free. Suppose therefore that the two generators are not distinct, so $\alpha \wedge c = \beta \wedge c$ in $F_{\mathbf{OML}}(\alpha, \beta)$. This would imply that the identity

$$\forall\, a, b \in L\colon\ (a \wedge b) \vee (a \wedge b') = (a \wedge b) \vee (a' \wedge b)$$

holds in every orthomodular lattice L. Suppose additionally that aCb, and therefore bCa; then the left-hand side reduces to a and the right one to b, so this identity would imply that in every orthomodular lattice, elements can only commute if they are equal. □

Let us summarize the results:

Corollary 3.1.8 (cf. [1], pp. 80–86; [6], p. 239)**.** *$F_{\mathbf{OML}}(\alpha, \beta)$ is isomorphic to $\mathbb{2}^4 \times \mathrm{MO}_2$; it is a finite orthomodular lattice with 96 elements.* □

3.2 Properties of the Two-Generated Free OML

The representation of $F_{\mathbf{OML}}(\alpha, \beta)$ as the direct product $\mathbb{2}^4 \times \mathrm{MO}_2$ of two elementary, well-understood lattices facilitates the analysis of many of its properties – particularly those which are defined by identities, since identities hold in $F_{\mathbf{OML}}(\alpha, \beta)$ iff they hold componentwise in the two factors $\mathbb{2}^4$ and MO_2. We begin by exposing a representation of the two generators α, β in the product $\mathbb{2}^4 \times \mathrm{MO}_2$. We first need an auxiliary result:

Theorem 3.2.1. *Let L be an ortholattice and $\{x_0, \ldots, x_{n-1}\}$ be a finite nonempty subset of L. Then it holds that*

$$[\![\{x_0, \ldots, x_{n-1}\}]\!] = [\{x_0, \ldots, x_{n-1}, x_0', \ldots, x_{n-1}'\}].$$

Proof. Essentially, we apply the first two steps of the transformation in the proof of Theorem 3.1.6. Let t be a term in the operations $\wedge$, $\vee$, $\mathbf{0}$, $\mathbf{1}$, $'$ and the variables $x_0, \ldots, x_{n-1}$. Eliminate $\mathbf{0}$ and $\mathbf{1}$ by replacing them with $x_0 \wedge x_0'$ and $x_0 \vee x_0'$; then, apply the De Morgan laws repeatedly to eliminate subterms of the form $(x \wedge y)'$ and $(x \vee y)'$; finally, apply $x'' = x$ repeatedly until there are no multiple applications of $'$. Now t is a term in the operations $\wedge$ and $\vee$ and the variables $x_0, \ldots, x_{n-1}$ plus their orthocomplements $x_0', \ldots, x_{n-1}'$. □

We use the above Theorem 3.2.1 to characterize the generators of $\mathbb{2}^4 \times \mathrm{MO}_2$. The obvious approach is somewhat problematic: simply showing that a pair of elements in $\mathbb{2}^4 \times \mathrm{MO}_2$ generates the two generators of $\mathbb{2}^4 \times \{\mathbf{0}\} \cong \mathbb{2}^4$ and the two generators of $\{\mathbf{0}\} \times \mathrm{MO}_2 \cong \mathrm{MO}_2$ is not enough, as the orthocomplementations in the two factors differ from the orthocomplementation on $\mathbb{2}^4 \times \mathrm{MO}_2$ restricted to the subsets $\mathbb{2}^4 \times \{\mathbf{0}\}$ and $\{\mathbf{0}\} \times \mathrm{MO}_2$ (which are not even closed under it).

However, the join and meet operations *do* coincide: therefore, by Theorem 3.2.1, it *does* suffice to generate the two generators of $\mathbb{2}^4$ *and their orthocomplements in* $\mathbb{2}^4$, plus the two generators of MO_2 and their orthocomplements in MO_2.

Theorem 3.2.2 (cf. [1], pp. 80–81; [6], p. 239)**.** *Denote an element of* $\mathbb{2}^4 \times \mathrm{MO}_2$ *by* (x, x_0, x_1, x_2, x_3)*, where* $x \in \mathrm{MO}_2$ *and* $x_0, x_1, x_2, x_3 \in \mathbb{2} = \{\mathbf{0}, \mathbf{1}\}$*. Let* $\mathtt{0}, \mathtt{a}, \mathtt{A}, \mathtt{b}, \mathtt{B}, \mathtt{1}$ *be the elements of* MO_2*. Then the following pair of elements generates* $\mathbb{2}^4 \times \mathrm{MO}_2$*:*

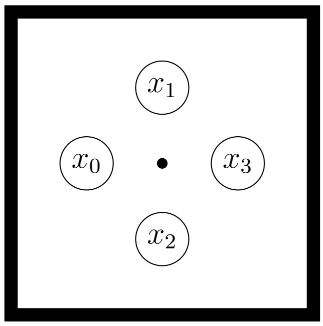
$$\alpha := (\mathtt{a}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}),$$
$$\beta := (\mathtt{b}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}).$$

Proof. In the proof of Theorem 3.1.6 we have seen that $(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1})$ generate $\mathbb{2}^4$; also, clearly, $\mathtt{a}, \mathtt{b}$ generate MO_2. Applying Theorem 3.2.1, it suffices to show that the following eight elements can all be generated by α, β:

$$\begin{aligned}
\alpha_0 &:= (\mathtt{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}) &&= (\alpha \wedge \beta) \vee (\alpha \wedge \beta'),\\
\beta_0 &:= (\mathtt{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}) &&= (\beta \wedge \alpha) \vee (\beta \wedge \alpha'),\\
\alpha_0' &:= (\mathtt{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}) &&= (\alpha' \wedge \beta) \vee (\alpha' \wedge \beta'),\\
\beta_0' &:= (\mathtt{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}) &&= (\beta' \wedge \alpha) \vee (\beta' \wedge \alpha'),\\
\alpha_1 &:= (\mathtt{a}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) &&= (\alpha' \vee \beta) \wedge (\alpha' \vee \beta') \wedge \alpha,\\
\beta_1 &:= (\mathtt{b}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) &&= (\beta' \vee \alpha) \wedge (\beta' \vee \alpha') \wedge \beta,\\
\alpha_1' &:= (\mathtt{A}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) &&= (\alpha \vee \beta) \wedge (\alpha \vee \beta') \wedge \alpha',\\
\beta_1' &:= (\mathtt{B}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) &&= (\beta \vee \alpha) \wedge (\beta \vee \alpha') \wedge \beta'.
\end{aligned}$$

The calculations are straightforward: in the Boolean part, any two elements commute, so the first four terms simplify to $\alpha, \beta, \alpha', \beta'$ and the last four to $\mathbf{0}$; in the MO_2 part, we have $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \vee \mathbf{b} = \mathbf{1}$ for all $\mathbf{a} \in \{\mathtt{a}, \mathtt{A}\}$ and $\mathbf{b} \in \{\mathtt{b}, \mathtt{B}\}$, so the first four terms simplify to $\mathbf{0}$ and the last four to $\alpha, \beta, \alpha', \beta'$. □

It quickly becomes apparent that the notation (x, x_0, x_1, x_2, x_3) for the elements of $F_{\mathbf{OML}}(\alpha, \beta)$ is rather verbose, and a more succinct notation is called for. To this end we present the notation proposed by Mirko Navara in [7], inspired by a set-theoretical representation. In this notation, every element of $F_{\mathbf{OML}}(\alpha, \beta)$ is represented by a symbol consisting of four dots which are each either hollow or filled (representing the $\mathbb{2}^4$ part) within the boundary of a square whose edges are either present or absent (representing the MO_2 part). The layout is clarified the figure below.



Navara symbols for elements of $F_{\mathbf{OML}}(\alpha, \beta)$

Each of the dots represents a separate factor of $\mathbb{2}$ within $\mathbb{2}^4$, where $\mathbf{0}$ is hollow and $\mathbf{1}$ is filled; the edges of the square represent an element of MO_2 by the following correspondence:

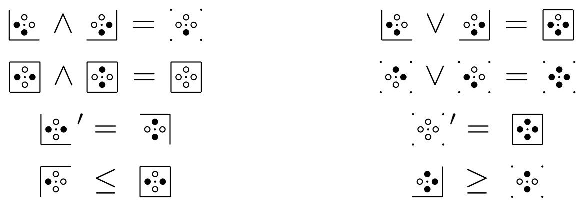
0	a	A	b	B	1

The meets and joins of each of these symbols are the set-theoretical intersections and unions of the edges, except that a single edge "degenerates" to the empty symbol, while a symbol with three edges "oversaturates" to the full square. Equivalently, the orthocomplementation coincides with set-theoretical complementation and the partial order coincides with set inclusion, but the joins and meets do not in general coincide with the set-theoretical unions and intersections, and have to be derived from the partial order (using the definition via infima and suprema).

The layout was chosen so that the zero, the unit, the generators α, β, and their orthocomplements have the symbols

$\mathbf{0}$	α	α'	β	β'	$\mathbf{1}$

which lend themselves well to the mnemonic that α is in the bottom left, β is in the bottom right, and orthocomplements are on opposite sides. (We identify the two generators α, β of $F_{\mathbf{OML}}(\alpha, \beta)$ with the two generators α, β of $\mathbb{2}^4 \times \mathrm{MO}_2$ from Theorem 3.2.2.) The meet, join, and orthocomplementation operations on these symbols are performed separately on the MO_2 part and on each of the four dots of the $\mathbb{2}^4$ part. Since the orthocomplementation coincides with set-theoretical complements on both the $\mathbb{2}^4$ part and the MO_2 part, the orthocomplement of a symbol is just its set-theoretical complement. Because the partial order on a lattice is defined through identities ($a \leq b$ iff $a = a \wedge b$ iff $b = a \vee b$), the partial order on a direct product of lattices is componentwise: if L_0 and L_1 are lattices, $a_0, b_0 \in L_0$ and $a_1, b_1 \in L_1$, then $(a_0, a_1) \leq (b_0, b_1)$ in $L_0 \times L_1$ iff $a_0 \leq b_0$ in L_0 and $a_1 \leq b_1$ in L_1. Therefore, the partial order is also indicated directly by the symbols: when one symbol is a subset of the other. Some examples of computation on these symbols are shown below.



Equipped with a compact notation for the elements of $F_{\mathbf{OML}}(\alpha, \beta)$, we now turn to investigating its properties. It will become apparent that in many cases the Navara symbols spell out statements about $F_{\mathbf{OML}}(\alpha, \beta)$ all by themselves, when they would not have been so clear using the conventional representation as terms in α and β.

We begin by characterizing the covering relation on $F_{\mathbf{OML}}(\alpha, \beta)$. Recall that for two elements x, y of a poset P, y *covers* x, in notation $x \prec y$, if $x < y$ and no $z \in P$ satisfies $x < z < y$. If L_0 and L_1 are two lattices, then from the componentwise definition of the partial order in the direct product of $L_0 \times L_1$ it follows that $(x_0, x_1) \prec (y_0, y_1)$ iff either $x_0 \prec y_0$ and $x_1 = y_1$, or $x_1 \prec y_1$ and $x_0 = y_0$. With this remark and by examining the Hasse diagram of MO_2, the following theorem is trivial:

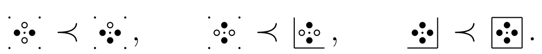
Theorem 3.2.3.

(i) In $F_{\mathbf{OML}}(\alpha, \beta)$, y covers x iff either the Boolean part of y covers the Boolean part of x and their MO_2 *parts coincide, or the* MO_2 *part of y covers the* MO_2 *part of x and their Boolean parts coincide.*

(ii) In $\mathfrak{2}^4$, (x_0, x_1, x_2, x_3) covers (y_0, y_1, y_2, y_3) iff exactly one $i \in \{0, 1, 2, 3\}$ satisfies $y_i = \mathbf{1}$, $x_i = \mathbf{0}$, and $\forall j \neq i\colon\ x_j = y_j$.

(iii) In MO_2*, $x \prec y$ iff either $y = \mathtt{1}, x \in \{\mathtt{a}, \mathtt{A}, \mathtt{b}, \mathtt{B}\}$, or $x = \mathtt{0}, y \in \{\mathtt{a}, \mathtt{A}, \mathtt{b}, \mathtt{B}\}$.* □

Note that in Navara's notation, $a \prec b$ iff the symbol of b has exactly one more part than the symbol for a. For instance,



The covering relation allows us to enumerate the atoms of $F_{\mathbf{OML}}(\alpha, \beta)$:

Theorem 3.2.4. *$F_{\mathbf{OML}}(\alpha, \beta)$ has eight atoms. Every element of $F_{\mathbf{OML}}(\alpha, \beta)$ is a join of these atoms; in other words, $F_{\mathbf{OML}}(\alpha, \beta)$ is an atomistic lattice.*

Proof. From the componentwise definition of the partial order it follows that an atom of $L_0 \times L_1$, where L_0 and L_1 are bounded lattices, is either of the form $(x_0, \mathbf{0})$ where x_0 is an atom of L_0, or $(\mathbf{0}, x_1)$ where x_1 is an atom of L_1. The disjoint union of these two sets forms the set of atoms of $L_0 \times L_1$. Let us therefore find the atoms of $\mathbb{2}^4$ and MO_2:

- $\mathbb{2}^4$: We have seen in the proof of Theorem 3.1.6 that $\mathbb{2}^4 = F_{\mathbf{Bool}}(\alpha, \beta)$ has four atoms: $\alpha \wedge \beta$, $\alpha \wedge \beta'$, $\alpha' \wedge \beta$, $\alpha' \wedge \beta'$.
- MO_2: From the Hasse diagram it is clear that its atoms are $\mathtt{a}, \mathtt{b}, \mathtt{A}, \mathtt{B}$.

In Navara's notation, the eight atoms of $F_{\mathbf{OML}}(\alpha, \beta)$ are:

that is, precisely the eight symbols with exactly one part. It is clear that by taking joins of these symbols, any arbitrary symbol can be constructed, thus $F_{\mathbf{OML}}(\alpha, \beta)$ is atomistic. (In fact, every orthomodular lattice is atomistic.) □

We next turn to commutativity in $F_{\mathbf{OML}}(\alpha, \beta)$. Since commutativity is defined by an identity, commutativity in a direct product of ortholattices is componentwise. We will see that the Navara symbols give a clear indication of when two given elements of $F_{\mathbf{OML}}(\alpha, \beta)$ commute, as well a simple identification of its central elements:

Theorem 3.2.5. *In $F_{\mathbf{OML}}(\alpha, \beta)$, two elements commute iff their MO_2 parts commute. The commutativity relation on MO_2 is minimal, i.e. only those pairs of elements that must commute in every ortholattice actually do commute: if one of them is the zero or the unit, or they are equal, or they are mutual orthocomplements.*

Proof. Since the other factor of $F_{\mathbf{OML}}(\alpha, \beta)$, namely $\mathbb{2}^4$, is a Boolean algebra, every pair of $\mathbb{2}^4$ parts commutes. Thus, the condition that both parts commute is equivalent to just the condition that the MO_2 parts commute. Clearly, if they are equal, or they are orthocomplements, or one of them is $\mathtt{0}$ or $\mathtt{1}$, they commute; in every other case it can be seen that the term $(x \wedge y) \vee (x \wedge y')$ will evaluate to $\mathtt{0}$ rather than x (see Table 3.1).

In Navara's notation, two elements commute iff their join and meet would be purely set-theoretical, i.e. we wouldn't need to apply the rules of degenerating and oversaturating for the MO_2 parts. □

Theorem 3.2.6. *The central elements of $F_{\mathbf{OML}}(\alpha, \beta)$ are precisely those whose MO_2 part is either $\mathtt{0}$ or $\mathtt{1}$. The center $C\big(F_{\mathbf{OML}}(\alpha, \beta)\big)$ (i.e. the set of all central elements) is isomorphic to $\mathbb{2}^4 \times \{\mathtt{0}, \mathtt{1}\} \cong \mathbb{2}^5$; it has 32 elements.*

Proof. Since commutativity in $F_{\mathbf{OML}}(\alpha, \beta)$ is componentwise, $C\big(F_{\mathbf{OML}}(\alpha, \beta)\big)$ is isomorphic to $C(\mathbb{2}^4) \times C(\mathrm{MO}_2)$. Since $\mathbb{2}^4$ is Boolean, it is equal to its own center. MO_2 has a trivial center: its zero and unit are obviously central, and every other element has an element that does not commute with it, so $C(\mathrm{MO}_2) = \{\mathtt{0}, \mathtt{1}\}$. □

Remark: MO_2 is rather special in having a minimal commutativity relation. An ortholattice L with minimal commutativity is one where $C = R$, where the relation R is defined by xRy iff $x = y$ or $x = y'$ or one of x, y is $\mathbf{0}$ or $\mathbf{1}$; if elements of the ortholattice are listed with $\mathbf{0}, \mathbf{1}$ at the extremes and orthocomplements on opposite

C	⸬	└	┘	┌	┐	□
⸬	✓	✓	✓	✓	✓	✓
└	✓	✓			✓	✓
┘	✓		✓	✓		✓
┌	✓		✓	✓		✓
┐	✓	✓			✓	✓
□	✓	✓	✓	✓	✓	✓

Table 3.1: The commutativity relation on MO_2

sides, then the table of R is shaped like the symbol "⊠". Clearly, R is a symmetric relation; thus, any ortholattices where $C = R$ holds must be orthomodular. Since $x \leq y$ implies xCy for all $x, y \in L$, no two non-zero, non-unit elements may be comparable; thus, the only possibilities are the one-element lattice, $\mathbb{2}$, and the orthomodular lattices MO_n for $n \geq 1$, consisting of $\mathbf{0}$, $\mathbf{1}$, and the $2n$ mutually incomparable elements $x_0, \ldots, x_{n-1}, x'_0, \ldots, x'_{n-1}$, all of which are simultaneously atoms and co-atoms.

We now turn our attention to congruence relations, or equivalently quotient algebras, of $F_{\mathbf{OML}}(\alpha, \beta)$. It is a known result in the theory of orthomodular lattices that congruence relations on an orthomodular lattice correspond bijectively to so-called p-ideals (see [6], p. 73). For a lattice L, a (lattice) *ideal* I is a sublattice that is additionally closed under meets with arbitrary elements of L, i.e. for any $x \in I, y \in L$, we have $x \wedge y \in I$. (The dual concept is that of a *filter*.) A *p-ideal* of a bounded lattice is an ideal that is additionally closed under perspectivity, so that for any $x \in I$ and $y \sim x$ we have $y \in I$. (Recall that two elements x, y of a bounded lattice L are *perspective* if they have a common complement, i.e. if there is $z \in L$ such that $x \wedge z = \mathbf{0} = y \wedge z$ and $x \vee z = \mathbf{1} = y \vee z$.) The p-ideals of an orthomodular lattice L are precisely the equivalence classes of $\mathbf{0}$ under the congruence relations on L, and these equivalence classes fully characterize the congruence relations ([6], p. 76–77; [2]). It therefore suffices to find the p-ideals of $F_{\mathbf{OML}}(\alpha, \beta)$. To this end we recall a few standard results about lattice ideals ([4], pp. 32–33).

- A lattice ideal I is equivalently a sublattice that is also a *down-set*, that is a set "closed under" $\leq$: for any $x \in I, y \in L$, $y \leq x$ implies $y \in I$. For if $x \in I, y \in L$ and $y \leq x$ holds, then $y = y \wedge x$, so y is the meet of two elements of which one is in I, which by the definition of an ideal implies $y \in I$. Conversely, if I is a sublattice and a down-set, and $x \in I$ and $y \in L$ are given, then $x \wedge y \leq x$, so by the definition of a down-set, $x \wedge y \in I$.

- The intersection of any set of ideals of a lattice L is again an ideal of L. Thus, it makes sense to define the *ideal generated* by a subset $A \subseteq L$, written $(A]$, as the intersection of all ideals of L which are supersets of A. For the ideal generated by a singleton whose only element is a, called a *principal ideal*, we also write $(a]$. (The notations $[A)$ and $[a)$ are used for filters.)
- For all $a \in L$, we have $(a] = \{x \wedge a \mid x \in L\} = \{x \in L \mid x \leq a\}$.
- For all $A \subseteq L$, we have $(A] = \{x \in L \mid \exists F \subseteq A, F \text{ finite nonempty}, x \leq \bigvee F\}$.

If $A \subseteq L$ is finite and nonempty, we get $(A] = \{x \in L \mid x \leq \bigvee A\}$. If L is additionally bounded, we also have $(A] = [\mathbf{0}, \bigvee A]$. Taking $A = \{a\}$ for arbitrary $a \in L$, we find that every interval of the form $[\mathbf{0}, a]$ is an ideal of L. If L is finite, then conversely, every ideal I of L is of the form $[\mathbf{0}, a]$ for some $a \in L$, namely $a = \bigvee I$. The inclusion $[\mathbf{0}, \bigvee I] \supseteq I$ is clear, since for all $x \in I$ it holds that $x \leq \bigvee I$. For the reverse inclusion, take $x \leq \bigvee I$; since I is a sublattice and L is finite, $\bigvee I \in I$; since I is a down-set, we conclude $x \in I$. Thus, in a finite lattice, there is a one-to-one correspondence between the ideals and the elements.

Since $F_{\mathbf{OML}}(\alpha, \beta)$ is a finite lattice, its ideals are precisely the intervals $[\mathbf{0}, x]$ for all choices of $x \in F_{\mathbf{OML}}(\alpha, \beta)$. The question remains which of them are closed under perspectivity. We therefore characterize the perspectivity relation on $F_{\mathbf{OML}}(\alpha, \beta)$. Since perspectivity is defined by identities, perspectivity in a direct product is also componentwise: $(a_0, a_1) \sim (b_0, b_1)$ in $L_0 \times L_1$ with common complement (c_0, c_1) iff $a_0 \sim b_0$ in L_0 with common complement c_0 and $a_1 \sim b_1$ in L_1 with common complement c_1. Therefore it suffices to characterize perspectivity in $\mathbb{2}^4$ and MO_2 separately.

Theorem 3.2.7. *In $F_{\mathbf{OML}}(\alpha, \beta)$, two elements are perspective iff their Boolean parts are equal and their* MO_2 *parts are perspective. Two elements of* MO_2 *are perspective iff they are equal or they are both not the zero and not the unit.*

Proof. Since $\mathbb{2}^4$ is a Boolean algebra, by Corollary 2.2.4, the $\mathbb{2}^4$ parts are perspective iff they are equal. For the MO_2 part, clearly two equal elements are perspective, and any two distinct, non-zero, non-unit elements are perspective, since in the set $\{\mathtt{a}, \mathtt{A}, \mathtt{b}, \mathtt{B}\}$, the meet of any two distinct elements is $\mathtt{0}$ and their join is $\mathtt{1}$.

To show these are all the perspective pairs, we show that in every bounded lattice, no non-zero element can be perspective to $\mathbf{0}$; then, dually, no non-unit can be perspective to $\mathbf{1}$ and the perspectivity relation on MO_2 will be fully characterized (see Table 3.2). Since a complement c of $\mathbf{0}$ must satisfy $\mathbf{1} = \mathbf{0} \vee c = c$, $\mathbf{1}$ is the only complement of $\mathbf{0}$. However, if $\mathbf{1}$ is a complement of any other element x, then $\mathbf{0} = \mathbf{1} \wedge x = x$.

In Navara's notation, two elements are perspective iff their Boolean parts coincide and their MO_2 parts have equal numbers of edges. □

Remarks:

- Since pairs of elements of MO_2 are perspective whenever it is not the case that in *every* bounded lattice they would not be perspective, the perspectivity

~	⁘	└	┘	┌	┐	□
⁘	✓					
└		✓	✓	✓	✓	
┘		✓	✓	✓	✓	
┌		✓	✓	✓	✓	
┐		✓	✓	✓	✓	
□						✓

Table 3.2: The perspectivity relation on MO_2

relation on MO_2 is maximal in the same way as its commutativity relation is minimal. The perspectivity relation is maximal if it is equal to the relation R defined on a bounded lattice by xRy iff $x = y$ or neither of x, y is $\mathbf{0}$ or $\mathbf{1}$. Thus, the two parts of $F_{\mathbf{OML}}(\alpha, \beta)$ are "counterparts" in several ways: $\mathbb{2}^4$ has maximal commutativity and minimal perspectivity, while MO_2 has minimal commutativity and maximal perspectivity.

- Maximal perspectivity is more common than minimal commutativity. Any bounded lattice L consisting of "parallel" chains (i.e. L has a family of sublattices C_i, all of which are chains, such that $\bigcup C_i = L$ and for any $i \neq j$ we have $C_i \cap C_j = \{\mathbf{0}, \mathbf{1}\}$) is maximally perspective if the number of chains is at least 3 and the length of each chain is at least 3. (In that case, pick any two non-zero, non-unit elements: if they are both in the same chain, then any element in a different chain is a common complement; if they are in two different chains, then any element in yet another chain is a common complement.)

We can now specify the p-ideals, and thus the congruence relations and quotient algebras, of $F_{\mathbf{OML}}(\alpha, \beta)$. For two algebras A_0, A_1 of the same type, with relations $\approx_0$ and $\approx_1$ respectively, we define their *product* $(\approx) = (\approx_0) \times (\approx_1)$ on $A_0 \times A_1$ by $(a_0, a_1) \approx (b_0, b_1)$ iff $a_0 \approx_0 b_0$ and $a_1 \approx_1 b_1$. It is an elementary computation that if $\approx_0$ and $\approx_1$ are congruence relations on A_0 and A_1 respectively, then their product is again a congruence relation on $A_0 \times A_1$. In the theorem below we see that all congruence relations on $F_{\mathbf{OML}}(\alpha, \beta)$ are of this form, i.e. products of congruence relations on its five simple lattice factors.

Theorem 3.2.8. *$F_{\mathbf{OML}}(\alpha, \beta)$ has 32 p-ideals, namely all the intervals of the form $[\mathbf{0}, c]$ where the MO_2 part of c is either $\mathtt{0}$ or $\mathtt{1}$, i.e. whenever c is central. These correspond one-to-one with 32 congruence relations on $F_{\mathbf{OML}}(\alpha, \beta)$. Every congruence relation on $F_{\mathbf{OML}}(\alpha, \beta)$ is a product of congruence relations on its five simple lattice* *factors* $\mathrm{MO}_2 \times \mathbb{2} \times \mathbb{2} \times \mathbb{2} \times \mathbb{2}$. *Thus, every quotient algebra of* $F_{\mathbf{OML}}(\alpha, \beta)$ *is a product in which some or all of these five factors have been optionally factored out.*

Proof. We know every ideal of $F_{\mathbf{OML}}(\alpha, \beta)$ is an interval of the form $[\mathbf{0}, c]$ for some element c. The question remains which of these intervals are closed under perspectivity. We begin by finding the *perspective closure* of singletons $\{c\}$ for any element c, i.e. the smallest superset of $\{c\}$ that is closed under perspectivity. For $a \in F_{\mathbf{OML}}(\alpha, \beta)$ or $\mathbb{2}^4$, and $b \in F_{\mathbf{OML}}(\alpha, \beta)$ or MO_2, we write $a \times b$ for the element of $\mathbb{2}^4 \times \mathrm{MO}_2$ with the Boolean part of a and the MO_2 part of b.

- If the MO_2 part of c is either $\mathtt{0}$ or $\mathtt{1}$, then c is the only element perspective to c, so the perspective closure of $\{c\}$ is $\{c\}$.
- Otherwise, the elements perspective to c are $c \times \mathtt{a}, c \times \mathtt{A}, c \times \mathtt{b}, c \times \mathtt{B}$ (one of which is c itself), so the perspective closure of $\{c\}$ consists of these four elements.

Recall that the partial order on $\mathbb{2}^4 \times \mathrm{MO}_2$ is componentwise. Thus, any interval $[\mathbf{0}, c]$ consists of those elements whose Boolean part is less than or equal to the Boolean part of c and whose MO_2 part is less than or equal to the MO_2 part of c.

- If the MO_2 part of c is $\mathtt{0}$, then $[\mathbf{0}, c]$ contains only elements with zero MO_2 part; all of them are perspective only to themselves, so $[\mathbf{0}, c]$ is a p-ideal.
- If the MO_2 part of c is $\mathtt{1}$, then $[\mathbf{0}, c]$ contains precisely those elements whose Boolean parts are less than or equal to the Boolean part of c; the MO_2 part does not matter. In that case, for any element $d \in [\mathbf{0}, c]$ with MO_2 part in $\{\mathtt{a}, \mathtt{A}, \mathtt{b}, \mathtt{B}\}$, we have that $d \times \mathtt{a}, d \times \mathtt{A}, d \times \mathtt{b}, d \times \mathtt{B}$ are also all in $[\mathbf{0}, c]$, so it is a p-ideal.
- If the MO_2 part of c is in $\{\mathtt{a}, \mathtt{A}, \mathtt{b}, \mathtt{B}\}$, then $[\mathbf{0}, c]$ will only contain elements whose MO_2 part is either zero or coincides with that of c. This excludes in particular the three other elements perspective to c, so in this case $[\mathbf{0}, c]$ is not a p-ideal.

Thus, $[\mathbf{0}, c]$ is a p-ideal iff the MO_2 part of c is zero or full, which is the case iff c is a central element. Each of these p-ideals corresponds uniquely to a congruence relation θ, with the p-ideal being the equivalence class of $\mathbf{0}$ under θ. (If the MO_2 part of c is full, then $\mathtt{1}$ and $\mathtt{0}$ are identified, so the MO_2 part of elements vanishes under θ; otherwise, it remains distinguished. If any of the four "dots" x_0, x_1, x_2, x_3 of c are "filled", i.e. having the value $\mathbf{1}$ in the corresponding $\mathbb{2}$ factor, then that factor vanishes under θ, otherwise, it remains distinguished.)

There are at least 32 congruence relations on $F_{\mathbf{OML}}(\alpha, \beta) \cong \mathrm{MO}_2 \times \mathbb{2} \times \mathbb{2} \times \mathbb{2} \times \mathbb{2}$, namely the 2^5 possible products of congruence relations on the five simple factors (choosing either the identity relation or the universal relation on each). Since there are 32 p-ideals, and thus exactly 32 congruence relations, on $F_{\mathbf{OML}}(\alpha, \beta)$, these are all its congruence relations.

The statement about quotient algebras follows immediately from the one-to-one correspondence between quotient algebras and congruence relations, since if one of the factors of a congruence relation on $F_{\mathbf{OML}}(\alpha, \beta)$ is the universal relation, then the corresponding factor gets collapsed into the one-element lattice, which is equivalent to removing it from the product. □

We finish this section with a brief remark on modularity.

Theorem 3.2.9 ([1], p. 86)**.** $F_{\mathbf{OML}}(\alpha, \beta)$ *is modular.*

Proof. Since modularity is defined by identities, the direct product of modular lattices is modular, thus the statement will be proved once we show that the two factors of $F_{\mathbf{OML}}(\alpha, \beta)$ are both modular.

- $\mathbb{2}^4$ is Boolean, thus distributive, which implies modularity.
- MO_2 is also modular; its Hasse diagram does not contain a pentagon. For an alternative proof, note that any interval $[x, y] \subseteq \mathrm{MO}_2$ has $\mathtt{0}$ as its lower endpoint, or $\mathtt{1}$ as its upper endpoint, or is a one-element set; in each case, the upper and lower contractions $x{\uparrow}y$ and $x{\downarrow}y$ clearly coincide.

□

Corollary 3.2.10. *In the variety of ortholattices there does not exist an identity in two variables characterizing modularity.*

Proof. If such an identity existed, then, since $F_{\mathbf{OML}}(\alpha, \beta)$ is modular, this identity would hold in $F_{\mathbf{OML}}(\alpha, \beta)$, and therefore, since it has two variables, in every orthomodular lattice; so this would imply that every orthomodular lattice is modular, in contradiction to the lattice of Figure 2.1. □

Chapter 4

Orthomodular Lattice Arithmetic

4.1 Triplex Symbols

We begin this section by recalling some basic facts from universal algebra. Let $\mathbf{K}$ be a variety of type τ, and let $n \in \mathbb{N}$; then we denote by $T_\tau(n)$ the algebra of *terms* in n variables and the algebraic operations of τ, and we denote by $T_{\mathbf{K}}(n)$ the quotient algebra of $T_\tau(n)$ under the congruence relation

$$s \approx t \quad \Longleftrightarrow \quad \text{the identity } s = t \text{ holds in every algebra of } \mathbf{K}$$

(equivalently: on every algebra of $\mathbf{K}$, the term functions induced by s and t coincide). It is easy to verify that this indeed defines a congruence relation on $T_\tau(n)$, and hence the algebraic operations are well-defined on $T_{\mathbf{K}}(n)$. We will call the elements of $T_{\mathbf{K}}(n)$ *n-ary term functions* on the variety $\mathbf{K}$. Strictly speaking this is incorrect; they are not functions, as they lack a domain and a codomain. The name "term functions on $\mathbf{K}$" is motivated by the fact that two terms which are equal in $T_{\mathbf{K}}(n)$ induce equal functions on every algebra of $\mathbf{K}$, and conversely, if two terms have the property that for every algebra of $\mathbf{K}$, they induce equal functions, then they are equal in $T_{\mathbf{K}}(n)$.

It is well-known that $T_{\mathbf{K}}(n)$ is isomorphic to $F_{\mathbf{K}}(n)$, the free algebra over $\mathbf{K}$ with n generators; in fact, $T_{\mathbf{K}}(n)$ is a canonical construction of $F_{\mathbf{K}}(n)$ (see [5], p. 65). The details of this result are summarized in the following theorem.

Theorem 4.1.1 (cf. [5], p. 65)**.** *The algebra $T_{\mathbf{K}}(n)$ of term functions in the variables $x_0, x_1, \ldots, x_{n-1}$ is isomorphic to the free algebra $F_{\mathbf{K}}(n)$ with n generators $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. Specifically, let $\phi : \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \to T_{\mathbf{K}}(n)$ be defined by $\phi(\alpha_i) = x_i$ for all i; then $T_{\mathbf{K}}(n)$ and $F_{\mathbf{K}}(n)$ are isomorphic via the mapping ε and its inverse η:*

$$\begin{aligned} \varepsilon : \ \ T_{\mathbf{K}}(n) \to F_{\mathbf{K}}(n) : \ \ \varepsilon(t) &= \hat{t}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}); \\ \eta : \ \ F_{\mathbf{K}}(n) \to T_{\mathbf{K}}(n) : \ \ \eta &= \overline{\phi}, \end{aligned}$$

where $\hat{t}$ is the term function induced by t on $F_{\mathbf{K}}(n)$, and $\overline{\phi}$ is the unique homomorphic extension of ϕ.

Proof.

- ε is injective: Let s, t be n-ary term functions such that $\varepsilon(s) = \varepsilon(t)$, that is,

$$\hat{s}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) = \hat{t}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}).$$

  Let A be any algebra in $\mathbf{K}$ and ϕ be any map $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \to A$. Then its unique homomorphic extension $\overline{\phi}$ maps the elements $\hat{s}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ and $\hat{t}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ to the same element of A, since they are equal. Thus the identity $s = t$ holds in A; since A was arbitrary, the identity $s = t$ holds in every algebra in $\mathbf{K}$, so $s = t$ in $T_{\mathbf{K}}(n)$.

- ε is surjective: Any element a of $F_{\mathbf{K}}(n)$ is given by some term function t in the n generators; thus $\varepsilon(t) = a$.

- $\eta \circ \varepsilon = \mathrm{id}_{T_{\mathbf{K}}(n)}$: For all $t \in T_{\mathbf{K}}(n)$ we have

$$\begin{aligned}
\eta(\varepsilon(t)) &= \eta(\hat{t}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})) \\
&= \overline{\phi}(\hat{t}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})) \\
&= \hat{t}(\phi(\alpha_0), \phi(\alpha_1), \ldots, \phi(\alpha_{n-1})) \\
&= \hat{t}(x_0, x_1, \ldots, x_{n-1}) \\
&= t.
\end{aligned}$$

□

On any algebra A of $\mathbf{K}$, any element of $T_{\mathbf{K}}(n)$ induces a function in n arguments. Through an appropriate definition of the algebraic operations on these functions, this "function-inducing" map – denoted in the previous theorem by $\hat{}$ – can be made into a homomorphism of $T_{\mathbf{K}}(n)$ onto the set of term functions on A in n arguments; this set, denoted by $T_A(n)$, consists of true functions. The next theorem describes this result in detail.

Theorem 4.1.2. *Let $\mathbf{K}$ be a variety of type τ, and let A be an algebra in $\mathbf{K}$. For all algebraic operators with symbol f and arity k in τ, define the operation f on $T_A(n)$ by*

$$\begin{aligned}
&\big(f(g_0, g_1, \ldots, g_{k-1})\big)(x_0, x_1, \ldots, x_{n-1}) \\
&\quad = f\big(g_0(x_0, x_1, \ldots, x_{n-1}), g_1(x_0, x_1, \ldots, x_{n-1}), \ldots, g_{k-1}(x_0, x_1, \ldots, x_{n-1})\big).
\end{aligned}$$

With these operations, $\hat{} : T_{\mathbf{K}}(n) \to T_A(n)$ is a surjective homomorphism, and therefore $T_A(n)$, being a homomorphic image of $T_{\mathbf{K}}(n)$, is itself an algebra of the variety $\mathbf{K}$.

Proof. The surjectivity of $\hat{}$ is obvious, since elements of $T_A(n)$ are by definition functions induced by some element of $T_{\mathbf{K}}(n)$. We verify the homomorphism property. Let therefore $t_0, \ldots, t_{k-1}$ be k elements of $T_{\mathbf{K}}(n)$, let f be an algebraic operation with arity k in $T_A(n)$, and set $t := f(t_0, t_1, \ldots, t_{k-1})$.

Let $a_0, \ldots, a_{n-1}$ be n elements of A. Then we obtain

$$\begin{aligned}
&\hat{t}(a_0, a_1, \ldots, a_{n-1}) \\
&= f\big(\hat{t}_0(a_0, a_1, \ldots, a_{n-1}), \hat{t}_1(a_0, a_1, \ldots, a_{n-1}), \ldots, \hat{t}_{k-1}(a_0, a_1, \ldots, a_{n-1})\big) \\
&= \big(f(\hat{t}_0, \hat{t}_1, \ldots, \hat{t}_{k-1})\big)(a_0, a_1, \ldots, a_{n-1}).
\end{aligned}$$

Since this holds for arbitrary $a_0, \ldots, a_{n-1} \in A$, we have $\hat{t} = f(\hat{t}_0, \hat{t}_1, \ldots, \hat{t}_{k-1})$. □

In the sequel we apply these results mostly to the case $n = 2$ and $\mathbf{K} = \mathbf{OML}$, the variety of orthomodular lattices. In this case it means that $F_{\mathbf{OML}}(\alpha, \beta)$ is isomorphic to $T_{\mathbf{OML}}(2)$, the algebra of *binary term functions on orthomodular lattices*, and the set of induced binary functions on any specific orthomodular lattice is a homomorphic image of $T_{\mathbf{OML}}(2)$ and thus of $F_{\mathbf{OML}}(\alpha, \beta)$. Thus, there are only 96 distinct binary term functions on orthomodular lattices, corresponding one-to-one with the 96 elements of $F_{\mathbf{OML}}(\alpha, \beta)$. For any orthomodular lattice L, the binary term functions induced on L by $T_{\mathbf{OML}}(2)$ are themselves given lattice-theoretical operations, defined for all $f, g \in T_L(2)$ by

- $x\ (f \wedge g)\ y := (x\ f\ y) \wedge (x\ g\ y);$ (T1)
- $x\ (f \vee g)\ y := (x\ f\ y) \vee (x\ g\ y);$ (T2)
- $x\ (f')\ y := (x\ f\ y)';$ (T3)
- $x\ \mathbf{0}\ y := \mathbf{0}; \quad x\ \mathbf{1}\ y := \mathbf{1}.$ (T4)

Theorem 4.1.2 immediately guarantees that under these operations, $T_L(2)$ is itself an orthomodular lattice.

A pair of generators of $T_{\mathbf{OML}}(2)$ is given by the images of α and β under η, which are the *left projection* $\triangleleft$ and the *right projection* $\triangleright$, defined by

$$x \triangleleft y = x, \qquad x \triangleright y = y.$$

Similar results also hold, for example, in the variety of Boolean algebras. Since $F_{\mathbf{Bool}}(2)$ has only $16 = 2^{2^2}$ elements, there are only 16 distinct binary term functions on Boolean algebras, and applying the definitions (T1)–(T4) to binary term functions induced on any Boolean algebra B makes $T_B(2)$ itself into a Boolean algebra. Interestingly, $16 = 2^{2^2}$ is also equal to the total number of binary *functions* (without assuming that they are term functions) on $\mathbb{2}$, which, after checking that all binary term functions on $\mathbb{2}$ are distinct, implies that every binary function on $\mathbb{2}$ is a term function. Together with the characterization of any Boolean algebra as a subalgebra of a direct product of copies of $\mathbb{2}$, this implies that any term function on Boolean algebras can be characterized by picking a binary function on $\mathbb{2}$ and applying it componentwise. Of course, this result has no analogue for orthomodular lattices.

Returning to the variety of orthomodular lattices, we find that the isomorphism of $F_{\mathbf{OML}}(\alpha, \beta)$ with $T_{\mathbf{OML}}(2)$ unlocks a new meaning of Navara's symbols: they can not only represent each of the 96 elements of $F_{\mathbf{OML}}(\alpha, \beta)$, but they also serve as a compact notation of any binary term function on orthomodular lattices, by identifying $\triangleleft$ with α and $\triangleright$ with β. Below are a few examples of binary term functions and the corresponding Navara symbols.

$\triangleleft$	$\triangleright$	$\wedge$	$\vee$
			
$\Downarrow$	$\Uparrow$	$\mathbf{0}$	$\mathbf{1}$
			

(Here $\mathbf{0}$ and $\mathbf{1}$ are defined as in (T4), that is, functions that ignore both of their arguments and return $\mathbf{0}$ and $\mathbf{1}$ respectively.)

Thanks to the isomorphism, one can apply lattice-theoretical operations to these symbols under their interpretation as binary functions (by the definitions (T1)–(T4)), and they follow the same computation rules also under the new interpretation. For example, on the one hand, the equation  $\wedge$  $=$  means that the equation $(\alpha \Downarrow \beta) \wedge (\alpha \Uparrow \beta) = \mathbf{0}$ holds for the two generators α, β of $F_{\mathbf{OML}}(\alpha, \beta)$; but on the other hand, it also means that the identity $(x \Downarrow y) \wedge (x \Uparrow y) = \mathbf{0}$ holds in every orthomodular lattice L, which can also be symbolically expressed as "$\Downarrow \wedge \Uparrow = \mathbf{0}$".

As an example to motivate the upcoming steps, one could write the definition of the commutativity relation as follows:

aCb iff $a = (a$  $b) \vee (a$  $b)$.

However, expressing it in this way leaves much to be desired: the notation is hardly more compact than the traditional $(a \wedge b) \vee (a \wedge b')$. The power of Navara symbols is that the orthocomplements, meets, and joins of any binary term functions are once again binary term functions, and Navara symbols can notate any binary term function on orthomodular lattices in just one symbol. In our case, the definition of commutativity can be condensed to

aCb iff $a = a$  b.

The elegance in this is that the correct symbol () for the binary term function $(a, b) \mapsto (a \wedge b) \vee (a \wedge b')$ was computed simply as  $\vee$ , by the same evaluation rules as when the Navara symbols stood for elements of $F_{\mathbf{OML}}(\alpha, \beta)$. Quite literally, the intermediate step was

aCb iff $a = a$ ( $\vee$ ) b,

(applying (T2)), and then evaluating the join of the symbols.

If the occurrences of the arguments a and b are not perfectly aligned for the application of (T1)–(T4), we can use a trick employing the left and right projections $\lhd$ and $\rhd$, with respective symbols ⌊⁘ and ⁘⌋. For example, one way to state the orthomodular law is

$$a \leq b \quad \Rightarrow \quad a \vee (a \text{ ⁘ } b) = b,$$

in which there are too many occurrences of a to apply (T2), so we replace the first a by $a \lhd b = a$ ⌊⁘ b. Now (T2) can be applied and we obtain the compact formulation

$$a \leq b \quad \Rightarrow \quad a \text{ ⌊⁘ } b = b.$$

In fact, the Navara symbols for the meet and join operations can be derived in this way: we rewrite

$$a \wedge b = (a \lhd b) \wedge (a \rhd b) = (a \text{ ⌊⁘ } b) \wedge (a \text{ ⁘⌋ } b) = a \ (\text{⌊⁘} \wedge \text{⁘⌋})\ b = a \text{ ⁘ } b,$$

and likewise for $\vee$. This gives rise to the very elegant mnemonic that the Navara symbols for the meet and join operations are the literal meet and join of the symbols for the generators, ⌊⁘ and ⁘⌋.

This process also generalizes: clearly, in any term t, replacing every occurrence of a by $a \lhd b$ and every occurrence of b by $a \rhd b$ creates a term in which all occurrences of a and b are "balanced" for the application of (T1)–(T4), so that replacing $\lhd$ and $\rhd$ with their Navara symbols, ⌊⁘ and ⁘⌋, allows to reduce the term to the form $a\ \Phi\ b$, where Φ is a single Navara symbol. By the isomorphism of $F_{\mathbf{OML}}(\alpha, \beta)$ with $T_{\mathbf{OML}}(2)$, we find that Φ encodes the same binary term function as t. We can also skip introducing the variables a, b and applying the axioms (T1)–(T4), at which point the first transformation we had applied to t becomes a substitution of ⌊⁘ for a and ⁘⌋ for b. In conclusion, the Navara symbol corresponding to the binary term function $t(a, b)$ is simply $t($⌊⁘ , ⁘⌋$)$.

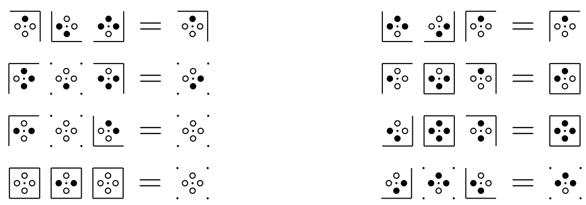
We can take this concept further, beyond just the lattice-theoretical operations. Since the elements of $F_{\mathbf{OML}}(\alpha, \beta) \cong T_{\mathbf{OML}}(2)$, represented by Navara symbols, can induce binary functions on any orthomodular lattice L, they can in particular induce binary functions on $T_{\mathbf{OML}}(2)$ itself. Therefore its elements can be thought of as binary functions from itself onto itself, which allows us to interpret a sequence $f\,g\,h$ of three elements of $F_{\mathbf{OML}}(\alpha, \beta)$ as evaluating to the single element $\hat{g}(f, h)$. We define this notion more precisely below.

In this context, by a ***symbol*** Φ we mean an element of $F_{\mathbf{OML}}(\alpha, \beta) \cong T_{\mathbf{OML}}(2)$, which we will usually write using Navara's notation. A sequence of three symbols, $\Phi\ \Xi\ \Psi$, will be called a ***triplex symbol***; it can be ***evaluated*** into a single element of $F_{\mathbf{OML}}(\alpha, \beta)$ by setting

$$\Phi\ \Xi\ \Psi := \hat{\xi}(\Phi, \Psi) \quad \text{where} \quad \xi := \eta(\Xi).$$

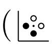
Here Ξ is seen as an element of $T_{F_{\mathbf{OML}}(\alpha,\beta)}(2)$, while Φ and Ψ are seen as elements of either $T_{\mathbf{OML}}(2)$ or $F_{\mathbf{OML}}(\alpha, \beta)$ interchangeably.

Thus, for an orthomodular lattice L, triplex symbols extend the concept of equipping $T_L(2)$ with lattice-theoretical operations into equipping it with all of $T_{\mathbf{OML}}(2)$; they generalize the computation on Navara symbols with the operations $\{\wedge, \vee, '\}$ to cases where the operation is itself an arbitrary Navara symbol. Some examples of triplex symbols along with their evaluations are presented below.



The first three rows represent computations which are already familiar; from top to bottom and left to right, in row-major order, the central symbols in these computations stand for the operations $\triangleleft, \triangleright, \wedge, \vee, \mathbf{0}, \mathbf{1}$. The computations in the bottom row are more complex, and it is not immediately clear how to arrive at their respective results. We can verify by inspection that the terms corresponding to the middle symbols are

a  $b = (a \vee b) \wedge (a' \vee b')$; a  $b = (a \wedge b') \vee (a' \wedge b) \vee (a' \wedge b')$,

and confirm the computations this way, but this offers no general procedure. While we know how to transform a given term t in two variables and the lattice-theoretical operations into the corresponding Navara symbol (just evaluate $t($, $)$), the *reverse* process – constructing a term that corresponds to a given Navara symbol, and thus evaluating any binary term function encoded as a Navara symbol – is non-trivial. We will abandon this approach, in favor of a method of evaluating triplex symbols that completely bypasses the explicit construction of a corresponding term.

These two final examples serve as motivation to find a procedure for automated evaluation of triplex symbols, as each of them demonstrates a potential application. We will see that the evaluation of triplex symbols can be used to characterize relations on orthomodular lattices defined by identities, and to compute simple variations of a binary term function (such as swapping the order of its arguments). In particular, we will see, among other things, that two elements $\Phi, \Psi \in T_{\mathbf{OML}}(2)$ are equal iff the triplex symbol Φ  Ψ equals $\mathbf{0}$, and that a binary term function $\Phi \in T_{\mathbf{OML}}(2)$ is commutative iff the triplex symbol  Φ  equals Φ. Therefore, these last two computations encode the fact that  = , and that the operation  is commutative, purely in terms of evaluation of triplex symbols.

We therefore turn our attention to the question of how to evaluate any given triplex symbol. The answer we present in the next section is applicable to both mental arithmetic and computer programs, demonstrated with an implementation in the programming language Haskell.

4.2 Evaluation of Triplex Symbols

At the heart of the techniques in this section is the following elementary observation, which is a special case of Theorem 4.1.2 and which we restate here for future reference:

Composition Rule for Symbols. *For any orthomodular lattice L and any $a, b \in L$, and for all symbols $\Phi, \Psi \in F_{\mathbf{OML}}(\alpha, \beta)$, we have*

$$a\ (\Phi \wedge \Psi)\ b = (a\ \Phi\ b) \wedge (a\ \Psi\ b); \qquad a\ (\Phi \vee \Psi)\ b = (a\ \Phi\ b) \vee (a\ \Psi\ b).$$

The core idea is to use this rule, along with the observation from Theorem 3.2.4 that $F_{\mathbf{OML}}(\alpha, \beta)$ is atomistic, to decompose any Navara symbol into its atomic components. Evaluating a triplex symbol $\Phi\ \Xi\ \Psi$ then reduces to evaluating each $\Phi\ \Xi_i\ \Psi$ for all i taken from some index set I, where the Ξ_i are all atoms of $F_{\mathbf{OML}}(\alpha, \beta)$, and then taking the join of them all. Since $F_{\mathbf{OML}}(\alpha, \beta)$ only has eight atoms, this reduces the number of base cases for Ξ from 96 to only 8, and any computation for an arbitrary Ξ will take at most 8 steps. Actually, we can do even better: Since the MO_2 part of Ξ always takes at most two atoms to assemble, and the only choice that requires two atoms is ⁙, we can include ⁙ as a base case: then, we have 9 base cases, but every computation takes only at most 5 steps, at most one of which is MO_2 and the remaining ones are all Boolean.

Thus it suffices to characterize the atomic binary term functions. The four Boolean atoms are each a single meet of at most two orthocomplements, analogous to ⁙ being the symbol for the meet operation itself; meanwhile, Theorem 3.2.2 already gives term representations for the four MO_2 atoms:

- a ⁙ $b \;=\; a \wedge b$,
- a ⁙ $b \;=\; a \wedge b'$,
- a ⁙ $b \;=\; a' \wedge b$,
- a ⁙ $b \;=\; a' \wedge b'$,
- a ⁙ $b \;=\; (a' \vee b) \wedge (a' \vee b') \wedge a$,
- a ⁙ $b \;=\; (b' \vee a) \wedge (b' \vee a') \wedge b$,
- a ⁙ $b \;=\; (a \vee b) \wedge (a \vee b') \wedge a'$,
- a ⁙ $b \;=\; (b \vee a) \wedge (b \vee a') \wedge b'$.

Finally, ⁙ is the symbol for the upper commutator:

$$a \text{ ⁙ } b \;=\; (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b') = a \Uparrow b.$$

The correctness of these terms can be verified by term substitution. Illustratively we verify ⁙:

(⁙ ∨ ⁙) ∧ (⁙ ∨ ⁙) ∧ ⁙ = (⁙ ∧ ⁙) ∧ ⁙ = ⁙ ∧ ⁙ = ⁙ .

This already suffices to construct an algorithm that can evaluate $a\ \Phi\ b$ for any symbol Φ and a, b taken from any orthomodular lattice L on which all the lattice-theoretical operations are defined: decompose Φ into its Boolean atoms and its MO_2 part, evaluate all the corresponding terms in a and b, and take the join. In particular, this algorithm is sufficient for evaluating triplex symbols.

However, it falls short of the goal of providing mental arithmetic: while the Boolean atoms are each just a single meet of possibly orthocomplemented inputs, the MO_2 atoms are significantly more complex. In fact, when specializing just for triplex symbols (that is, assuming $L = F_{\mathbf{OML}}(\alpha, \beta)$), we can do much better: we can simplify the computation of the MO_2 atoms drastically.

Note, firstly, what happens when a and b commute. In that case $a\ \boxed{\substack{\circ\\ \circ\cdot\circ\\ \circ}}\ b = \mathbf{0}$ by Theorem 3.1.1. This implies $a\ \Phi\ b = \mathbf{0}$ for any "purely MO_2" symbol Φ (that is, a symbol whose Boolean parts are all zero), because such a symbol Φ satisfies $\Phi \leq \boxed{\substack{\circ\\ \circ\cdot\circ\\ \circ}}$, thus $\boxed{\substack{\circ\\ \circ\cdot\circ\\ \circ}} = \Phi \vee \boxed{\substack{\circ\\ \circ\cdot\circ\\ \circ}}$, and therefore

$$\mathbf{0} = a\ \boxed{\substack{\circ\\ \circ\cdot\circ\\ \circ}}\ b = a\,(\Phi \vee \boxed{\substack{\circ\\ \circ\cdot\circ\\ \circ}})\,b = (a\ \Phi\ b) \vee (a\ \boxed{\substack{\circ\\ \circ\cdot\circ\\ \circ}}\ b) = (a\ \Phi\ b) \vee \mathbf{0} = a\ \Phi\ b.$$

Thus, for all five MO_2 base cases, the value is simply $\mathbf{0}$ if the two arguments commute. In particular, when evaluating triplex symbols, the Boolean part of the result of any purely MO_2 operation is always $\mathbf{0}$, since all pairs of elements commute in a Boolean algebra. Therefore, the result is always *itself* purely MO_2, and depends only on the MO_2 components of the operands.

As a result, we can shortcut the five base cases by working out their operation tables in MO_2. Out of the 36 possible pairs of MO_2 elements, 28 commute, and thus will result in `0` for each of the five operations; the remaining 8 pairs must be checked manually for each operation to identify further possibilities for shortcuts. We accomplish this with a Haskell program.

We begin by defining ortholattices: the meet and join are binary operations, the orthocomplement is unary and the zero and unit are nullary. The meet and join, and the zero and unit, can also be defined in terms of each other using De Morgan's laws.

```haskell
infixr 5 ∧, ∨     -- ∧ and ∨ are right associative with precedence 5

class Ortholattice t where
  (∧) :: t → t → t       -- meet
  (∨) :: t → t → t       -- join
  cpl  :: t → t           -- orthocomplement
  zero :: t               -- least element
  unit :: t               -- greatest element

  a ∧ b = cpl (cpl a ∨ cpl b)
  a ∨ b = cpl (cpl a ∧ cpl b)
  zero  = cpl unit
  unit  = cpl zero
```

Next, we define the operations we wish to tabulate on MO_2. They include the four atomic MO_2 operations and the (non-atomic) upper commutator.

```
mo2l   :: (Ortholattice t) ⇒ t → t → t
mo2r   :: (Ortholattice t) ⇒ t → t → t
mo2l'  :: (Ortholattice t) ⇒ t → t → t
mo2r'  :: (Ortholattice t) ⇒ t → t → t
upcom  :: (Ortholattice t) ⇒ t → t → t

a `mo2l`  b  = let a' = cpl a; b' = cpl b in a  ∧ (a'∨ b ) ∧ (a'∨ b')
a `mo2r`  b  = let a' = cpl a; b' = cpl b in b  ∧ (a ∨ b') ∧ (a'∨ b')
a `mo2l'` b  = let a' = cpl a; b' = cpl b in a' ∧ (a ∨ b ) ∧ (a ∨ b')
a `mo2r'` b  = let a' = cpl a; b' = cpl b in b' ∧ (a ∨ b ) ∧ (a'∨ b )
a `upcom` b  = let a' = cpl a; b' = cpl b in (a∨b) ∧ (a∨ b') ∧ (a'∨ b) ∧ (a'∨ b')
```

Now we can make MO_2 into an ortholattice, by defining its ortholattice operations:

```
data MO2 = O | A | B | B' | A' | I  deriving (Eq, Enum, Bounded)

instance Ortholattice MO2 where

  O ∧ _            = O    -- 0 ∧ x = 0
  _ ∧ O            = O    -- x ∧ 0 = 0
  I ∧ x            = x    -- 1 ∧ x = x
  x ∧ I            = x    -- x ∧ 1 = x
  x ∧ y | x == y   = x    -- x ∧ x = x
  _ ∧ _            = O    -- x ∧ y = 0 otherwise

  I ∨ _            = I    -- 1 ∨ x = 1
  _ ∨ I            = I    -- x ∨ 1 = 1
  O ∨ x            = x    -- 0 ∨ x = x
  x ∨ O            = x    -- x ∨ 0 = x
  x ∨ y | x == y   = x    -- x ∨ x = x
  _ ∨ _            = I    -- x ∨ y = 1 otherwise

  cpl O   = I
  cpl A   = A'
  cpl B   = B'
  cpl B'  = B
  cpl A'  = A
  cpl I   = O

  zero = O
  unit = I
```

Tabulating the five purely MO_2 operations on MO_2 is now a straightforward matter (see e.g. the function *drawTable* in the full source code). The results are presented below, where the rows correspond to the left argument and the columns to the right argument. The rows and columns for `0` and `1` have been omitted in each case, because – as remarked earlier – the result is always `0` if either argument is equal to one of them, since in these cases the two arguments commute.

└⁘	└	┘	┌	┐
└	∷	└	└	∷
┘	┘	∷	∷	┘
┌	┌	∷	∷	┌
┐	∷	┐	┐	∷

⁘┘	└	┘	┌	┐
└	∷	┘	┌	∷
┘	└	∷	∷	┐
┌	└	∷	∷	┐
┐	∷	┘	┌	∷

⁘┐	└	┘	┌	┐
└	∷	┐	┐	∷
┘	┌	∷	∷	┌
┌	┘	∷	∷	┘
┐	∷	└	└	∷

┌⁘	└	┘	┌	┐
└	∷	┌	┘	∷
┘	┐	∷	∷	└
┌	┐	∷	∷	└
┐	∷	┌	┘	∷

Table 4.1: The four atomic MO_2 operations on MO_2

□⁘	└	┘	┌	┐
└	∷	□	□	∷
┘	□	∷	∷	□
┌	□	∷	∷	□
┐	∷	□	□	∷

Table 4.2: The upper commutator operation on MO_2

As we can see by inspecting these tables, each of the five purely MO_2 operations can be reduced to a simple rule if the operands are themselves both elements of MO_2: For any purely MO_2 operation Φ and any $a, b \in MO_2$, if aCb, then $a \; \Phi \; b = 0$; otherwise,

$$a \;\lfloor⁘\; b = a; \qquad a \;⁘\rfloor\; b = b; \qquad a \;\overline{⁘|}\; b = a'; \qquad a \;\overline{|⁘}\; b = b'; \qquad a \;\boxed{⁘}\; b = 1.$$

Given that commutativity in MO_2 is very straightforward to test for, this gives us a much more efficient method to compute the purely MO_2 operations with MO_2 operands (and, therefore, also with $F_{\mathbf{OML}}(\alpha, \beta)$ operands), completing the need for a better algorithm for evaluating triplex symbols. We now proceed to implement it.

In preparation for making $F_{\mathbf{OML}}(\alpha, \beta)$ into an ortholattice, we first make $\mathbb{2}$ into one. In Haskell, $\mathbb{2}$ is represented by the type $Bool = \{\mathsf{False}, \mathsf{True}\}$, with the operations of logical conjunction, disjunction, and negation:

```
instance Ortholattice Bool where
  (∧) = (&&)    -- logical conjunction
  (∨) = (||)    -- logical disjunction
  cpl  = not    -- logical negation
  zero = False
  unit = True
```

Now we can make $F_{\mathbf{OML}}(\alpha, \beta)$ (here called *F2OML*) into an ortholattice, via the direct product $MO_2 \times \mathbb{2} \times \mathbb{2} \times \mathbb{2} \times \mathbb{2}$. The operations are all simply componentwise.

```
data F2OML = F2 MO2 Bool Bool Bool Bool

instance Ortholattice F2OML where

  F2 a al au ad ar ∧ F2 b bl bu bd br = F2 (a∧b) (al∧bl) (au∧bu) (ad∧bd) (ar∧br)
  F2 a al au ad ar ∨ F2 b bl bu bd br = F2 (a∨b) (al∨bl) (au∨bu) (ad∨bd) (ar∨br)

  cpl (F2 x xl xu xd xr) = F2 (cpl x) (cpl xl) (cpl xu) (cpl xd) (cpl xr)

  zero = F2 zero zero zero zero zero
  unit = F2 unit unit unit unit unit
```

Finally, we can implement the triplex symbol evaluation algorithm. We break up the middle symbol into its atomic components, where $\boxed{⁘}$ is considered atomic. The Boolean parts are evaluated directly. The MO_2 parts each always produce a result whose Boolean part is zero and whose MO_2 part is determined by the rules from above: if the arguments commute, it is zero, otherwise they return the left or right argument, or their complements, or the unit, respectively.

Commutativity in $F_{\mathbf{OML}}(\alpha, \beta)$ was characterized in Theorem 3.2.5: we ignore the Boolean parts and test only if the MO_2 parts commute, which do so minimally.

```
comF2OML :: F2OML → F2OML → Bool
(F2 a _ _ _ _) comF2OML (F2 b _ _ _ _) -- extract just the MO2 parts
  | a = zero    = True     -- 0Cb
  | b = zero    = True     -- aC0
  | a = unit    = True     -- 1Cb
  | b = unit    = True     -- aC1
  | a = b       = True     -- aCa
  | a = cpl b   = True     -- aCa′
  | otherwise   = False    -- ¬(aCb) otherwise
```

We next define the auxiliary function *projectMO2*, the "projection" onto the MO_2 component: it takes an element of $F_{\mathbf{OML}}(\alpha, \beta)$ and sets all its Boolean parts to zero.

```
projectMO2 :: F2OML → F2OML
projectMO2 (F2 x _ _ _ _) = F2 x zero zero zero zero
```

Next, we define *evaluateMO2*, our "shortcut" for how to evaluate the six purely MO_2 operations (encoded here as elements of MO_2) for elements of $F_{\mathbf{OML}}(\alpha, \beta)$.

```
evaluateMO2 :: F2OML → MO2 → F2OML → F2OML
evaluateMO2 a _  b | a comF2OML b = zero   -- if aCb, the result is always zero
evaluateMO2 _ O  _                = zero
evaluateMO2 a A  _                = projectMO2 a          -- a b
evaluateMO2 _ B  b                = projectMO2 b          -- a b
evaluateMO2 _ B' b                = projectMO2 (cpl b)    -- a b
evaluateMO2 a A' _                = projectMO2 (cpl a)    -- a b
evaluateMO2 _ I  _                = projectMO2 unit       -- a b
```

Finally, we can evaluate triplex symbols:

```
evaluateTriplex :: F2OML → F2OML → F2OML → F2OML
evaluateTriplex a (F2 f fl fu fd fr) b = let
  a' = cpl a
  b' = cpl b
  xl = if fl then a ∧ b'  else zero     -- a b = a ∧ b′
  xu = if fu then a' ∧ b' else zero     -- a b = a′ ∧ b′
  xd = if fd then a ∧ b   else zero     -- a b = a ∧ b
  xr = if fr then a' ∧ b  else zero     -- a b = a′ ∧ b
  x  = evaluateMO2 a f b                -- evaluate the MO2 part
  in x ∨ xl ∨ xu ∨ xd ∨ xr              -- take the join of all results
```

(The above function *evaluateTriplex*, while functional, still lacks an interface: we can't easily specify triplex symbols to be evaluated nor print the results. This is implemented in the full source code by making *F2OML* an instance of **Show** and **Read**. This code is not shown here; we leave the details to the interested reader.)

Note that this improvement is specific to the evaluation of triplex symbols: the "simple" algorithm from the beginning of this section (just evaluating a term corresponding to each atomic component) *cannot* be improved in a way analogous to *evaluateTriplex* if a and b are taken to be elements of an arbitrary orthomodular lattice. This is because *projectMO2* is not a term function: there is no term t in one variable and the ortholattice operations which induces *projectMO2* on $F_{\mathbf{OML}}(\alpha, \beta)$. Such a term t would have to correspond to an element of $F_{\mathbf{OML}}(\alpha)$, the free orthomodular lattice with one generator; and it is easy to see that this lattice consists of the four elements $\mathbf{0}, \mathbf{1}, \alpha, \alpha'$, whose corresponding term functions are $x \mapsto \mathbf{0}, x \mapsto \mathbf{1}, \mathrm{id}, {}'$. (It is free over $\{\alpha\}$ also in the class of ortholattices and Boolean algebras.) Clearly, none of these term functions induce *projectMO2* on $F_{\mathbf{OML}}(\alpha, \beta)$.

To expose a simple mechanical application of our algorithm for evaluating triplex symbols, we begin by formulating a stronger version of the Composition Rule which holds for all triplex symbols:

Composition Rule for Triplex Symbols. *For any orthomodular lattice L and any $a, b \in L$, and for all symbols $\Phi, \Xi, \Psi \in F_{\mathbf{OML}}(\alpha, \beta)$, we have*

$$a\ (\Phi\ \Xi\ \Psi)\ b = (a\ \Phi\ b)\ \Xi\ (a\ \Psi\ b).$$

This version follows from the weaker one simply by noting that homomorphisms, which preserve all fundamental operations, necessarily also preserve all term functions. From this rule we can deduce the following "self-preserving" identity of triplex symbols:

$$\text{For all symbols } \Phi \in F_{\mathbf{OML}}(\alpha, \beta), \text{ we have } \ \text{⌊•∵}\ \Phi\ \text{∵•⌋} = \Phi.$$

Indeed,

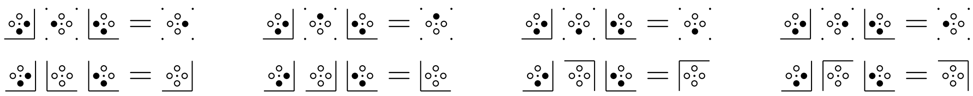
$$a\ (\text{⌊•∵}\ \Phi\ \text{∵•⌋})\ b = (a\ \text{⌊•∵}\ b)\ \Phi\ (a\ \text{∵•⌋}\ b) = (a \lhd b)\ \Phi\ (a \rhd b) = a\ \Phi\ b.$$

What happens if we swap the order of the arguments?

$$a\ (\text{∵•⌋}\ \Phi\ \text{⌊•∵})\ b = (a\ \text{∵•⌋}\ b)\ \Phi\ (a\ \text{⌊•∵}\ b) = (a \rhd b)\ \Phi\ (a \lhd b) = b\ \Phi\ a.$$

Thus the triplex symbol $\text{∵•⌋}\ \Phi\ \text{⌊•∵}$ produces the *converse function* $\tilde{\Phi}$, defined by $a\ \tilde{\Phi}\ b = b\ \Phi\ a$. Similar transformations are possible with related triplex symbols; for example, the triplex symbol $\text{∵•⌉}\ \Phi\ \text{⌈•∵}$ produces the function Φ^{C} defined by $a\ \Phi^{\mathsf{C}}\ b = a'\ \Phi\ b'$; we shall call it the *conjugate* of Φ.

A question arises whether Navara's notation simplifies the computation of such function transformations, perhaps into a geometric alteration of the corresponding symbol. Using our algorithm, we know there is no need to check all 96 symbols in an attempt to find a rule; it suffices to transform the eight atoms. To illustrate, we show the computations for producing converse functions:



Thus we see that every atom is mapped to its horizontal mirror image, and therefore all Navara symbols are mapped to their horizontal mirror images. Navara's notation thus elegantly allows to construct any given symbol's converse simply by flipping it horizontally.

Carrying out analogous computations for combinations of complementing the left argument, complementing the right argument, and interchanging the arguments reveals that each such combination corresponds to a geometric transformation of the Navara symbol, represented by an element of the dihedral group D_4:

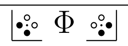
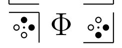
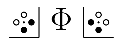
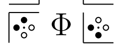
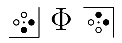
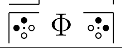
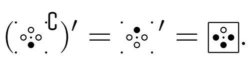
Triplex symbol	Function	Geometric transformation
	$a \ \Phi \ b$	Identity
	$a' \ \Phi \ b$	Reflection along the diagonal ╲
	$a \ \Phi \ b'$	Reflection along the diagonal ╱
	$a' \ \Phi \ b'$	Rotation by a half turn
	$b \ \Phi \ a$	Horizontal reflection
	$b' \ \Phi \ a$	Rotation by a quarter turn clockwise
	$b \ \Phi \ a'$	Rotation by a quarter turn counterclockwise
	$b' \ \Phi \ a'$	Vertical reflection

Table 4.3: Geometric transformations of Navara symbols

This can be used, for instance, to derive the *dual symbol* of any given Navara symbol Φ. The dual symbol to Φ is defined as the symbol Φ^δ such that $a \ \Phi^\delta \ b = (a' \ \Phi \ b')'$. It is so called because, for any orthomodular lattice L, the symbol corresponding to Φ in the dual lattice to L is Φ^δ. Since the dual is the complement of the conjugate, from the above table it follows that the dual of a Navara symbol is formed by rotating it by half a turn and building the set-theoretical complement (or doing those steps in the reverse order). For instance, the dual of  is



Indeed,  and  are the symbols for $\wedge$ and $\vee$ respectively, which are mutually dual.

Another use is an easy proof of a statement, appearing in [3], that out of the 96 binary term functions on orthomodular lattices, exactly 16 are commutative. A binary term function is commutative iff it is equal to its own converse, which, for a Navara symbol, means that it must be horizontally symmetric. Therefore, four independent binary choices are possible: the bottom dot, the top dot, the two side dots (either both present or both absent), and the MO_2 part (either zero or full), leading to 16 possibilities overall.

4.3 Application: Relations defined by identities

This section aims to expose a more advanced application of arithmetic on triplex symbols, to study so-called *relations defined by identities* on orthomodular lattices.

For a variety $\mathbf{K}$ of type τ, we shall call an element of $\big(T_\tau(2)\big)^2$ a *binary term identity*. Define Θ_τ as the set of finite subsets of $\big(T_\tau(2)\big)^2$. On any algebra A of $\mathbf{K}$, an element S of Θ_τ can *induce a binary relation* $\overline{S}$ as follows: for all $a, b \in A$,

$$a \ \overline{\big\{(s_0,t_0),(s_1,t_1),\ldots,(s_{n-1},t_{n-1})\big\}} \ b$$
$$\text{iff}$$
$$\forall\ i < n\colon\ s_i(a,b) = t_i(a,b).$$

Define $\Theta_{\mathbf{K}}$ as the quotient set of Θ_τ under the equivalence relation

$$S \approx S' \quad \Leftrightarrow \quad \text{for all algebras } A \in \mathbf{K},\ \overline{S} = \overline{S'} \text{ in } A$$

(that is, two elements of Θ_τ are equivalent iff on every algebra of $\mathbf{K}$, the binary relations induced by them coincide). In analogy to the beginning of this chapter, through abuse of terminology we will refer to elements of $\Theta_{\mathbf{K}}$ as *binary relations on* $\mathbf{K}$ *defined by identities*. To give a few examples, on any variety $\mathbf{K}$, the equality relation and the universal relation are both defined by identities, by taking $S = \{(\triangleleft, \triangleright)\}$ and $S = \varnothing$ respectively. The empty relation can never be defined by identities, since every variety contains one-element algebras, which satisfy every identity; however, there may exist a "near-empty" relation, which induces the empty relation on every nontrivial algebra of $\mathbf{K}$.

Some examples for the case $\mathbf{K} = \mathbf{OML}$:

- $\leq$ is defined by identities, since $a \leq b$ is defined by $a = a \wedge b$ (equivalently $b = a \vee b$), so we can take $S = \{(a, a \wedge b)\}$ or $S = \{(b, a \vee b)\}$.
- The *orthogonality* relation $\perp$, where $a \perp b$ is defined by $a \leq b'$ (equivalently $b \leq a'$). In general, if a relation R is defined by identities, then so is any relation of the form $s(a,b)\, R\, t(a,b)$, where $s, t \in T_{\mathbf{OML}}(2)$.
- The commutativity relation C, defined for example by $a = (a \wedge b) \vee (a \wedge b')$.
- An example defined by two identities is *position* P' (see e.g. [7]), using $S = \{(a \wedge b', \mathbf{0}), (a' \wedge b, \mathbf{0})\}$.
- *Non*-examples include $<$, $\prec$, or $\sim$ (perspectivity).

For orthomodular lattices, a special case of relations defined by identities are *relations defined by a symbol*, that is, relations R which admit a definition

$$aRb \quad \text{iff} \quad a\ \Phi\ b = \mathbf{0},$$

for some symbol $\Phi \in F_{\mathbf{OML}}(\alpha, \beta)$. In this context, we shall call Φ the *characteristic symbol* of R. Every such relation is defined by identities (since for every symbol $\Phi \in F_{\mathbf{OML}}(\alpha, \beta)$ there exists a term t that encodes the same binary function). Interestingly, the converse also holds: to every relation defined by identities on orthomodular lattices, there exists a characteristic symbol. We will prove this statement in two steps.

Firstly, any relation defined by *multiple* symbols can also be characterized by *one*:

Theorem 4.3.1. *Let $S \in \Theta_{\mathbf{OML}}$ be defined by the identities*

$$a \ \Phi_0 \ b = \mathbf{0}; \quad a \ \Phi_1 \ b = \mathbf{0}; \ \ldots \ a \ \Phi_{n-1} \ b = \mathbf{0}$$

where $\Phi_i \in F_{\mathbf{OML}}(\alpha, \beta)$ for all $i < n$, $n \in \mathbb{N}$. Then S can also be defined by a single identity $a \ \Phi \ b = \mathbf{0}$, namely with $\Phi = \Phi_0 \vee \Phi_1 \vee \ldots \vee \Phi_{n-1}$.

Proof. The statement is trivial for $n = 1$. For $n = 0$, the universal relation, ⁙ is a characteristic symbol (since a ⁙ $b = \mathbf{0}$ for all a, b in any orthomodular lattice), and ⁙ $= \bigvee \varnothing$. For $n > 1$, we will prove the case $n = 2$; the general case follows inductively. Suppose then that a relation $S \in \Theta_{\mathbf{OML}}$ is defined by aSb iff $a \ \Phi \ b = a \ \Psi \ b = \mathbf{0}$ for two symbols $\Phi, \Psi \in F_{\mathbf{OML}}(\alpha, \beta)$. We have seen in the first chapter that in any lattice L with zero, for all $x, y \in L$, ($x = \mathbf{0}$ and $y = \mathbf{0}$) is equivalent to $x \vee y = \mathbf{0}$; thus, the condition $a \ \Phi \ b = a \ \Psi \ b = \mathbf{0}$ is equivalent to $\mathbf{0} = (a \ \Phi \ b) \vee (a \ \Psi \ b) = a \ (\Phi \vee \Psi) \ b$. Therefore, $\Phi \vee \Psi$ is a characteristic symbol of S. □

(Note that an analogous statement with the meet of symbols characterizing logical disjunction of multiple identities does not hold, since $x \wedge y = \mathbf{0}$ does *not* imply ($x = \mathbf{0}$ or $y = \mathbf{0}$). The case $n = 0$ makes it particularly clear that this cannot be the case: the logical conjunction of zero identities gives the universal relation, which has a characteristic symbol; the logical *disjunction* of zero identities gives the empty relation, which cannot be defined by identities in any way, let alone by a symbol.)

The full equivalence between relations defined by identities and relations defined by a characteristic symbol will then be proved once we show that any relation defined by a *single* identity also admits a characterization through a symbol. To this end, we first prove a seemingly unrelated auxiliary result:

Theorem 4.3.2. *Let L be an orthomodular lattice and $a, b \in L$. The following statements are equivalent:*

(i) $a = b$;

(ii) a, b are in position P' and additionally aCb.

Proof. (i) $\Rightarrow$ (ii) is trivial, since commutativity is reflexive and for position P' we obtain $a \wedge b' = b \wedge b' = \mathbf{0}$ and $a' \wedge b = a' \wedge a = \mathbf{0}$. Suppose therefore that a, b are in position P' and aCb holds. Then we obtain $a = (a \wedge b) \vee (a \wedge b') = (a \wedge b) \vee \mathbf{0} = a \wedge b$, so that $a \leq b$. We also have bCa, since L is orthomodular, and therefore also $b = (b \wedge a) \vee (b \wedge a') = (b \wedge a) \vee \mathbf{0} = b \wedge a$, so $b \leq a$ as well. By antisymmetry, $a = b$, proving (ii) $\Rightarrow$ (i). □

The significance of this result lies in the fact that both commutativity and each of the two "components" of position P' have characteristic symbols: position P' can be expressed as $a \diamond b = \mathbf{0}$ and $a \diamond b = \mathbf{0}$, while we know from Theorem 3.1.1 that aCb holds iff $a \boxdot b = \mathbf{0}$. Thus, we can apply Theorem 4.3.1 and construct the join of these three symbols to obtain the following central result:

$\boxdot$ is the characteristic symbol of the equality relation.

Using triplex symbol arithmetic, this allows us to construct a characteristic symbol for any relation defined by a single identity:

Theorem 4.3.3. *Let $S \in \Theta_{\mathbf{OML}}$ be defined by a single identity $s(a, b) = t(a, b)$ for some terms $s, t \in T_{\mathbf{OML}}(2)$. Then there exists a symbol $\Xi \in F_{\mathbf{OML}}(\alpha, \beta)$ such that aSb iff $a\ \Xi\ b = \mathbf{0}$ holds (i.e. Ξ is a characteristic symbol of S), namely $\Xi = \Phi \boxdot \Psi$, where Φ and Ψ are the symbols corresponding to s and t respectively.*

Proof. We apply the composition rule for triplex symbols, along with the fact that $\boxdot$ characterizes equality:

$$\begin{aligned} aSb &\Leftrightarrow s(a, b) = t(a, b) \\ &\Leftrightarrow a\ \Phi\ b = a\ \Psi\ b \\ &\Leftrightarrow (a\ \Phi\ b) \boxdot (a\ \Psi\ b) = \mathbf{0} \\ &\Leftrightarrow a\ (\Phi \boxdot \Psi)\ b = \mathbf{0}. \end{aligned}$$

□

Thus in the case of a single identity, there always exists a characterization through a symbol. In the case of multiple identities, find the symbol for each one and take the join. (Since $F_{\mathbf{OML}}(\alpha, \beta)$ is a finite lattice, this even works for infinitely many identities.) Finally, in the case of zero identities, take $\diamond$ as the characteristic symbol. Overall, it follows that any relation characterized by a set of identities has a characterization by a single symbol; thus, to summarize:

Corollary 4.3.4. *In the variety of orthomodular lattices, the set of relations characterized by identities is identical to the set of relations characterized by a symbol.* □

In particular, the existence of a symbol for any relation characterized by identities is always guaranteed. What about uniqueness? To test this, let us construct the characteristic symbol of $a \leq b$ using its two equivalent definitions by identities:

$$a = a \wedge b \quad \Leftrightarrow \quad a \llcorner b = a \diamond b \quad \Leftrightarrow \quad a\ (\llcorner \boxdot \diamond)\ b = \mathbf{0} \quad \Leftrightarrow \quad a \llcorner b = \mathbf{0};$$

$$b = a \vee b \quad \Leftrightarrow \quad a \lrcorner b = a \boxdot b \quad \Leftrightarrow \quad a\ (\lrcorner \boxdot \boxdot)\ b = \mathbf{0} \quad \Leftrightarrow \quad a \ulcorner b = \mathbf{0}.$$

The two approaches gave different results. This is not a contradiction to Theorem 4.3.3: both $\llcorner$ and $\ulcorner$ are indeed characteristic symbols for $\leq$. Their term functions are not identical, but they do return $\mathbf{0}$ under the same conditions. Thus, uniqueness is not always given. To clarify exactly when it isn't, and by how much, we begin with the following observation:

Theorem 4.3.5. *Let L be an orthomodular lattice and $a, b \in L$. The following statements are equivalent:*

(i) aCb;

(ii) $a \boxed{⁘} b = \mathbf{0}$;

(iii) $a \ \Phi \ b = \mathbf{0}$ *for some nonzero, purely* MO_2 *symbol* Φ;

(iv) $a \ \Phi \ b = \mathbf{0}$ *for all nonzero, purely* MO_2 *symbols* Φ.

Proof. (i) $\Leftrightarrow$ (ii) has been established in Theorem 3.1.1. (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial.

To show (iii) $\Rightarrow$ (iv), one way is to inspect Tables 4.1 and 4.2: wherever any one of the five operations takes the value $\mathbf{0}$, all five do. Alternatively, avoiding the need for direct calculation, we can rephrase the statements in terms of quotient algebras: we consider $[\![\{a, b\}]\!]$, which, being generated by two elements, is isomorphic to a quotient algebra of $F_{\mathbf{OML}}(\alpha, \beta)$. We make use of Theorem 3.2.8 stating that all congruence relations on $F_{\mathbf{OML}}(\alpha, \beta)$ are products of congruence relations on its simple factors, along with Theorem 3.1.3 stating that MO_2 is a simple lattice. This lets us conclude that if a congruence relation on $F_{\mathbf{OML}}(\alpha, \beta)$ identifies $\mathbf{0}$ with some nonzero, purely MO_2 element, then it collapses the entire MO_2 part of $F_{\mathbf{OML}}(\alpha, \beta)$. $\square$

The above theorem implies that $\lfloor⁘$, $⁘\rfloor$, $⁘\rceil$, $\lceil⁘$, $\boxed{⁘}$ are all equivalent as characteristic symbols for the commutativity relation. Similarly, $\Phi \times \mathtt{a}, \Phi \times \mathtt{b}, \Phi \times \mathtt{A}, \Phi \times \mathtt{B}, \Phi \times \mathtt{1}$ are equivalent for any symbol Φ, and we pick the form $\Phi \times \mathtt{1}$ as the "standard form". Using $\lfloor$•⁘ and $\boxed{•⁘}$ as an example,

$$
\begin{aligned}
& a \ \lfloor•⁘ \ b = \mathbf{0} \\
\Leftrightarrow \ & (a \ \dot{•⁘} \ b) \vee (a \ \lfloor⁘ \ b) = \mathbf{0} \\
\Leftrightarrow \ & a \ \dot{•⁘} \ b = \mathbf{0} \text{ and } a \ \lfloor⁘ \ b = \mathbf{0} \\
\Leftrightarrow \ & a \ \dot{•⁘} \ b = \mathbf{0} \text{ and } a \ \boxed{⁘} \ b = \mathbf{0} \\
\Leftrightarrow \ & (a \ \dot{•⁘} \ b) \vee (a \ \boxed{⁘} \ b) = \mathbf{0} \\
\Leftrightarrow \ & a \ \boxed{•⁘} \ b = \mathbf{0}.
\end{aligned}
$$

Up to equivalence of standard forms, there are 32 distinct symbols in $F_{\mathbf{OML}}(\alpha, \beta)$; they form a subalgebra isomorphic to $2^4 \times \{\mathtt{0}, \mathtt{1}\} \cong 2^5$. The theorem below shows, among other things, that this is the only redundancy in characteristic symbols:

Theorem 4.3.6. *Two symbols $\Phi, \Psi \in F_{\mathbf{OML}}(\alpha, \beta)$ define the same relation in $\Theta_{\mathbf{OML}}$ iff Φ and Ψ have the same standard form. Thus, there are exactly 32 relations defined by identities on orthomodular lattices.*

Proof. There are at most 32 relations defined by identities, since 32 is the number of elements of $\mathbb{2}^4 \times \{\mathtt{0}, \mathtt{1}\}$. To show that there are no further redundancies, consider the five atoms of $\mathbb{2}^4 \times \{\mathtt{0}, \mathtt{1}\}$ (⁘, ⁘, ⁘, ⁘, ⊡) and the corresponding relations: $a \wedge b' = \mathbf{0}; a' \wedge b' = \mathbf{0}; a \wedge b = \mathbf{0}; a' \wedge b = \mathbf{0}; aCb$. (We will denote the first four by the symbols $\lesssim, \underset{\sim}{\top}, \underset{\sim}{\bot}, \gtrsim$ respectively.) Every relation defined by identities is some (possibly empty) conjunction of these conditions, thus we need to verify that no two such conjunctions are equivalent.

Firstly, let $n, m \in \mathbb{N}$, and let p_i and q_j be families of formulas in two bound variables a, b, and suppose that the following proposition holds:

$$\forall\, a, b\colon\ \big(\forall\, i < n\colon\ p_i(a,b)\big) \;\Leftrightarrow\; \big(\forall\, j < m\colon\ q_j(a,b)\big),$$

where the sets $\{p_i \mid i < n\}$ and $\{q_j \mid j < m\}$ are not equal. In that case, one of these sets contains a formula that the other set does not; suppose WLOG that $\{q_j \mid j < m\}$ contains such a formula, call it q. Then we conclude that the following must hold:

$$\forall\, a, b\colon\ \big(\forall\, i < n\colon\ p_i(a,b)\big) \;\Rightarrow\; q(a,b),$$

and the p_i for $i < n$ are all distinct from q.

If two different elements of $\mathbb{2}^4 \times \{\mathtt{0}, \mathtt{1}\}$ correspond to the same relation defined by identities, then there are two different sets of conjunctions of "atomic relations" equivalent to one another. Therefore, by the above, one of these atomic relations must be implied by the remaining four in every orthomodular lattice L and for all $a, b \in L$. To disprove this, for each atomic relation R we present an orthomodular lattice L and two elements $a, b \in L$ such that a, b are in every atomic relation other than R, but not R itself:

L	a	b	$\lesssim$	$\underset{\sim}{\top}$	$\underset{\sim}{\bot}$	$\gtrsim$	C
$\mathbb{2}$	$\mathbf{1}$	$\mathbf{0}$		✓	✓	✓	✓
$\mathbb{2}$	$\mathbf{0}$	$\mathbf{0}$	✓		✓	✓	✓
$\mathbb{2}$	$\mathbf{1}$	$\mathbf{1}$	✓	✓		✓	✓
$\mathbb{2}$	$\mathbf{0}$	$\mathbf{1}$	✓	✓	✓		✓
MO_2	$\mathtt{a}$	$\mathtt{b}$	✓	✓	✓	✓	

Table 4.4: Mutual independence of the five atomic relations

Thus, none of the atomic relations is implied by the remaining four, and so all 32 possible conjunctions of them are pairwise distinct. □

The statement of Theorem 4.3.6 exposes a surprisingly regular structure for relations defined by identities on orthomodular lattices. They form a lattice themselves, which is also isomorphic to $\mathbb{2}^5$. In this lattice, the meet is the set-theoretical intersection, though – as remarked earlier – the join is just a supremum, and generally not the set-theoretical union; for instance, in this lattice, $(\leq) \vee (\geq) = C \supsetneq (\leq) \cup (\geq)$. The unit of this lattice is the universal relation; the zero is the "near-empty" relation characterized by the symbol $\boxdot$ and the corresponding identity $\mathbf{1} = \mathbf{0}$, which is universally true in the one-element lattice and universally false in all others. The five so-called "atomic relations" are actually the *co*-atoms of this lattice, and every element is a (possibly empty) meet of them. Some example representations of relations as meets of co-atoms include:

- $(\leq) \;=\; (\lesssim) \wedge C, \quad (\geq) \;=\; (\gtrsim) \wedge C;$
- $(\perp) \;=\; (\underset{\sim}{\perp}) \wedge C;$
- $(\top) \;=\; (\underset{\sim}{\top}) \wedge C$, where $\top$ is the dual relation to $\perp$;
- $P' \;=\; (\lesssim) \wedge (\gtrsim);$
- $(=) \;=\; (\lesssim) \wedge (\gtrsim) \wedge C \;=\; (\leq) \wedge (\geq) \;=\; P' \wedge C.$

Relations such as $<$, $\prec$, or $\sim$ are also definable on any orthomodular lattice, but they do not admit characterizations through identities only. To prove this, it suffices to eliminate each atomic relation as a possible component. To illustrate, we close this section with a proof for perspectivity:

Corollary 4.3.7. *The perspectivity relation cannot be defined by identities only.*

Proof. We exclude all atomic relations R that *cannot* be part of a conjunction that assembles $\sim$, by finding an orthomodular lattice L and elements $a, b \in L$ such that $a \sim b$, but $\neg\,(aRb)$ holds:

- In $\mathbb{2}$, $\mathbf{1} \sim \mathbf{1}$ but $\neg\,(\mathbf{1} \underset{\sim}{\perp} \mathbf{1})$; likewise, $\mathbf{0} \sim \mathbf{0}$ but $\neg\,(\mathbf{0} \underset{\sim}{\top} \mathbf{0})$.
- In MO_2, $\mathtt{a} \sim \mathtt{b}$ but $\neg\,(\mathtt{a}\,C\,\mathtt{b})$.
- In the orthomodular lattice of Figure 2.1, we have $c' \sim a$, but $c' \wedge a' = b \neq \mathbf{0}$, so $\neg\,(c' \lesssim a)$. Likewise, $a \sim c'$ but $\neg\,(a \gtrsim c')$.

We have eliminated all five atomic relations. Thus the only possible conjunction for $\sim$ would be the empty conjunction; but $\sim$ is not the universal relation (e.g. $\neg\,(\mathbf{0} \sim \mathbf{1})$ holds in $\mathbb{2}$). Therefore, $\sim$ cannot be characterized by identities. $\square$

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Bibliography

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